# Quantum Matrix Models for Simple Current Orbifolds *

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Abstract: An algebraic formulation of the stringy geometry on simple current orbifolds of the WZW models of type $A_N$ is developed within the framework of Reflection Equation Algebras, $\mathrm{REA}_q(A_N)$. It is demonstrated that $\mathrm{REA}_q(A_N)$ has the same set of outer automorphisms as the corresponding current algebra $A_N^{(1)}$ which is crucial for the orbifold construction. The CFT monodromy charge is naturally identified within the algebraic framework. The ensuing orbifold matrix models are shown to yield results on brane tensions and the algebra of functions in agreement with the exact BCFT data.

Keywords: WZW models, simple current orbifolds, quantum groups, reflection equation algebras.

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## Contents

## 1. Introduction.

Physics of D-branes has long been a subject of intense study[1], driven by the motivation to obtain new insights into the structure of the moduli space of string theory proper and to better understand the emergence of an essentially stringy geometry and gauge dynamics within any field-theoretic or matrix model approach to the propagation of strings in curved gravitational backgrounds with fluxes. An example of such a background is a compact Lie group $G$ ([2] and [3]), or a quotient thereof, known to support a nontrivial Kalb–Ramond field in a conformally invariant theory. Maximally symmetric (or untwisted) D-branes on $G$ have been shown to localise stably ([4]) around a discrete set of conjugacy classes ([5]) and are enumerated by dominant integral affine weights $\Lambda$ from the fundamental affine alcove $P^\kappa_+(\mathfrak{g})$[2], i.e. we have:

$$\text{(untwisted) D-branes on } SU(N+1) \quad \sim \quad \mathcal{R}ep_{\substack{\textit{integrable}\\ \textit{highest weight}}}\left(A_N^{(1)}\right) := \bigoplus_{\Lambda\in P^\kappa_+(A_N)} R_\Lambda, \tag{1.1}$$

in the case of interest, with

$$P^\kappa_+(A_N) := \left\{ \Lambda = \sum_{i=1}^N \lambda_i \Lambda^i \in P^*(A_N) \quad \middle| \quad \forall_{i\in\overline{1,N}} \ : \ \lambda_i \in \mathbb{N} \ \ \wedge \ \ \sum_{i=1}^N \lambda_i \leq \kappa \right\}, \tag{1.2}$$

where $P^*(A_N)$ is the weight space of $A_N$ and $\Lambda^i$ are the fundamental weights. The above defines the particular class of WZW manifolds whose orbifolds shall be the subject matter of the present paper.

In [7] a (quantum) matrix model of branes in WZW models was constructed, based on certain quantum algebras called Reflection Equation (RE) algebras. The latter have representation theory closely related to that of $U_q(A_N)$, which - in turn - is known to have irreducible highest weight representations such as those of $A_N^{(1)}$.

In this paper we extend the former construction to a new class of WZW backgrounds, namely - the so-called simple current orbifolds of the WZW geometries of type $A_N$. Simple currents are known to form an Abelian group, contained as a proper subgroup in the group of outer automorphisms of $A_N^{(1)}$ (we shall denote the latter by $\mathcal{O}ut\left(A_N^{(1)}\right)$). Thus an orbifolding of the corresponding matrix model based on $\mathrm{REA}_q(A_N)$ requires that the algebra has the same group of outer automorphisms as the affine algebra. Indeed, we show that $\mathcal{O}ut\left(\mathrm{REA}_q(A_N)\right) \cong \mathcal{O}ut\left(A_N^{(1)}\right)$ and moreover, that the action of both sets of automorphisms on branes is identical. As a result, $\mathcal{O}ut\left(\mathrm{REA}_q(A_N)\right)$ acquires a geometrical meaning necessary to perform the construction of the orbifold. Furthermore, it is straightforward to identify the CFT monodromy charge in $\mathrm{REA}_q(A_N)$ so that the orbifolding itself admits a purely algebraic and hence natural realisation in the framework developed, reminiscent of the original approach to orbifold models advanced in [8].

[1]See [1] for a detailed review and an exhaustive list of references.

[2]Here, as anywhere else in this paper, we restrict our analysis of the pre-orbifolding CFT to the diagonal case in which Cardy's classification of boundary states applies ([6]).

At this stage we may already outline and assess the content of the present paper. In Sect.2. we briefly describe RE algebras, their automorphisms and their relation to branes, shifting some technicalities to the appendices. In the following section we apply our results in a construction of quantum orbifolds and discuss their properties in some detail, whereby we also arrive at a particularly straightforward interpretation of the monodromy projection (known from BCFT) accompanying simple current orbifolding. These two sections form the core of the paper. In Sect.4 we present an explicit example of branes on the $SU(2)/\mathbb{Z}_2$ orbifold.

The all-important details and notation can be found in the Appendices. The first one, App.A, introduces the necessary BCFT background. In App.B we present a path leading from BCFT to the quantum matrix model. The remaining Appendices contain some relevant information on the algebras: $\mathcal{U}_q^{\mathrm{ext}}(A_N)$ and $\mathrm{REA}_q(A_N)$ and their representations.

## 2. The Reflection Equation Algebras and branes.

Similarities in the representation theory of the algebras: $A_N^{(1)}$ and $\mathcal{U}_q(A_N)$ ($q$ being a root of unity) are widely appreciated. There are, however, important differences as well. One of them is the outer automorphisms group[3], $\mathcal{O}ut = \mathcal{A}ut/\mathcal{I}nt$. For $A_N^{(1)}$ the latter reflects the symmetries of the appropriate Dynkin diagram, $\mathcal{O}ut\left(A_N^{(1)}\right) = \mathbb{Z}_{N+1} \ltimes \mathbb{Z}_2$, while for $\mathcal{U}_q(A_N)$ we have $\mathcal{O}ut(\mathcal{U}_q(A_N)) = \mathbb{Z}_2^N \ltimes \mathbb{Z}_2$. The difference is crucial in the context of simple current orbifolding since - according to [9] - the group generated by simple currents under OPE is precisely the strictly affine factor of $\mathcal{O}ut\left(A_N^{(1)}\right)$, that is - $\mathbb{Z}_{N+1}$.

In this section, we investigate the so-called Reflection Equation Algebra ([10, 11]), closely related to a modification of $\mathcal{U}_q(A_N)$ named the extended quantum universal enveloping algebra and denoted as $\mathcal{U}_q^{\mathrm{ext}}(A_N)$. As we explore its representation theory it shall become clear that, beside representations, $\mathrm{REA}_q(A_N)$ shares with $A_N^{(1)}$ the set of outer automorphisms $\mathbb{Z}_{N+1} \ltimes \mathbb{Z}_2$, with - as indicated by our results - the same geometrical meaning of the $\mathbb{Z}_{N+1}$ factor as in the affine setup. The last property is crucial for constructing orbifold models based on $\mathrm{REA}_q(A_N)$ in strict analogy with the WZW orbifolds discussed in App.A.

Recall that the Reflection Equation Algebra, $\mathrm{REA}_q(A_N)$, is the algebra generated by the operator entries of the matrix $\mathbf{M}$ determined by the celebrated Reflection Equation ([10]):

$$\mathbf{R}_{12}\mathbf{M}_1\mathbf{R}_{21}\mathbf{M}_2 = \mathbf{M}_2\mathbf{R}_{12}\mathbf{M}_1\mathbf{R}_{21}. \tag{2.1}$$

The independent central terms of this algebra are given by ([12])

$$\mathfrak{c}_k := \mathrm{tr}_q(\mathbf{M}^k), \qquad k \in \overline{1, N}, \tag{2.2}$$

$$\mathcal{K} := M^{\kappa_N}_{N+1,N+1} = e^{-\frac{2\pi i}{N+1}\sum_{n=1}^{N} nH_n}, \tag{2.3}$$

[3] $\mathcal{A}ut$ is the set of all automorphisms, with the identity adjoined to endow it with the group structure. $\mathcal{I}nt$ is the set of inner automorphisms.

where on the r.h.s. of the last formula we used a specific representation of $M_{N+1,N+1}$, to be justified presently. We interpret $M_{ij}$'s as coordinate "functions" on a quantum manifold, thus $\mathfrak{c}_k$ represent algebraic constraints on positions of submanifolds to be associated with branes. Amazingly, the last Casimir, $\mathcal{K}$, is related to the BCFT monodromy charge as one can easily check by comparing the above with (A.17).

There is one more scalar, the quantum determinant, to be set to some specific value:

$$\det_q \mathbf{M} \sim \mathbb{I}. \tag{2.4}$$

As an immediate consequence of (2.1) and (2.4) we get $\mathbf{M} \longrightarrow e^{\frac{2\pi i L}{N+1}}\mathbf{M},\ L \in \mathbb{Z}_{N+1}\backslash\{0\}$ as outer automorphisms of $\mathrm{REA}_q(A_N)$. Here we shall be interested in finite dimensional representations of $\mathrm{REA}_q(A_N)$ induced through the homomorphisms: $\mathrm{REA}_q(A_N) \hookrightarrow \mathcal{U}_q^{\mathrm{ext}}(A_N) \hookrightarrow \mathcal{U}_h(A_N)$ from highest weight irreducible representations of the latter algebra, with a non-vanishing quantum dimension (see App. C.2). We shall demonstrate that the above homomorphisms give rise to $\mathcal{O}ut(\mathrm{REA}_q(A_N)) = \mathbb{Z}_{N+1} \ltimes \mathbb{Z}_2$, with the $\mathbb{Z}_{N+1}$ factor realised as above and the remaining $\mathbb{Z}_2$ being the standard mirror symmetry of the Dynkin diagram of $A_N$.

The action of the outer automorphisms (C.16)-(C.17) (with the generator denoted by $\eta$) reads

$$\eta : M_{ij} \to e^{\frac{2\pi i}{N+1}} M_{ij}, \tag{2.5}$$

which implies

$$\eta \ :\ \mathfrak{c}_k \to e^{\frac{2\pi i k}{N+1}} \mathfrak{c}_k \quad , \quad \mathcal{K} \to e^{\frac{2\pi i \kappa}{N+1}} \mathcal{K}. \tag{2.6}$$

It appears that the above is the same as the left action of the inverse of the Casimir (2.3),

$$\mathcal{K}^{-1} \triangleright_L M_{ij} := \pi_{ik}\left(\mathcal{K}^{-1}\right) M_{kj} = e^{\frac{2\pi i}{N+1}} M_{ij} \equiv \eta\left(M_{ij}\right). \tag{2.7}$$

This is just an algebraic analogue of the ordinary geometrical left regular action:

$$g \longrightarrow \gamma g, \qquad g \in SU(N+1)\ ,\ \gamma \in Z(SU(N+1)) \cong \mathbb{Z}_{N+1}. \tag{2.8}$$

Next, we turn to $\mathcal{O}ut(A_N^{(1)})$. To begin with, recall that $\mathbb{Z}_{N+1} \subset \mathcal{O}ut\left(A_N^{(1)}\right)$ can be identified with a (discrete rotational) symmetry of the fundamental affine alcove $P_+^{\kappa}(A_N)$ and also with a cyclic symmetry of the extended Dynkin diagram of $A_N^{(1)}$. Moreover, $\mathbb{Z}_{N+1}$ has a deep geometrical meaning: it is isomorphic to the centre of the group $SU(N+1)$, thus orbifolding by $\mathbb{Z}_{N+1}$ means taking the quotient $SU(N+1)/\mathbb{Z}_{N+1}$ with respect to the natural action (2.8). In the Dynkin basis, the action of its generator on $P_+^{\kappa}(A_N)$ takes the following form:

$$\omega_N^* \ :\ [\lambda_1, \lambda_2, \ldots, \lambda_N] \to \left[\lambda_2, \lambda_3, \ldots, \lambda_N, \kappa - \sum_{i=1} \lambda_i\right]. \tag{2.9}$$

It results in the rescaling of the eigenvalues of the scalars (2.2)-(2.3):

$$\mathfrak{c}_k \to e^{\frac{2\pi i k}{N+1}} \mathfrak{c}_k \quad , \quad k_{\epsilon_{N+1}}^{2\kappa_N} \to e^{\frac{2\pi i \kappa}{N+1}} k_{\epsilon_{N+1}}^{2\kappa_N}, \tag{2.10}$$

which matches (2.6) exactly. Thus the action of $\mathbb{Z}_{N+1} \subset \mathcal{O}ut\left(A_N^{(1)}\right)$ on $\mathfrak{c}_k$ is the same as that of $\mathbb{Z}_{N+1} \subset \mathcal{O}ut\,(\mathrm{REA}_q(A_N))$.

In order to directly relate both sets of automorphisms in a physically meaningful manner we need to modify our former definition of the brane in the algebraic setup. In [7], string theory branes in the $SU(N+1)$ group manifold were associated with irreducible representations of $\mathrm{REA}_q(A_N)$. Here we propose to take the specific $(N+1)$-fold direct sum of irreducible representations:

$$\mathcal{B}_\Lambda := \bigoplus_{L=0}^{N} R^L_{(\omega_N^*)^L\Lambda} \tag{2.11}$$

as describing the brane assigned to the weight $\Lambda$. The (quantum) manifold of $SU(N+1)$ is thus "foliated" by the set of branes corresponding to all $\Lambda \in P_+^\kappa(A_N)$. Notice that all the summands in (2.11) have the same Casimir eigenvalues which is necessary for the correct geometrical picture. The advantage of (2.11) is that[4]

$$\eta^*\mathcal{B}_\Lambda = \mathcal{B}_{\omega_N^*\Lambda}, \tag{2.12}$$

i.e. the corresponding branes are isomorphic (the same Casimir eigenvalues and the same algebra of functions). In this setting $\eta^*$ is indistinguishable from $\omega_N^*$ and so it acquires an analogous geometrical meaning, which shall be crucial in our subsequent considerations. In particular, it allows us to define the orbifold by dividing out the action of $\mathbb{Z}_{N+1}$ generated by $\eta$. Orbifolding will "glue" branes with $\omega_N^*$-conjugate Casimirs.

For some non-generic weights ("fixed point" weights) we may have $(\omega_N^*)^D\Lambda_{FP} = \Lambda_{FP}$ (with $\frac{N+1}{D} =: n \in \mathbb{N}$), i.e. $(\eta^*)^D\mathcal{B}_{\Lambda_{FP}} = \mathcal{B}_{\Lambda_{FP}}$. When evaluated on those "fixed point" weights $\Lambda_{FP}$, some of the Casimirs vanish. This will lead to certain new interesting phenomena discussed in Sec.3.2.

As an aside, we note here that as we introduce (2.11) the requirement of consistency with the BCFT data for the pre-orbifolding WZW models necessitates a reformulation of the original quantum matrix model, which we now take in the general form[5]:

$$S^{eff}_{q,A_N} := \frac{T_0}{N+1}\mathrm{Tr}_q\left[\mathbb{I} + \mathrm{cov}_{\mathcal{U}_q(A_N)}(\mathbf{M})\right]. \tag{2.13}$$

Here $\mathrm{cov}_{\mathcal{U}_q(A_N)}(\mathbf{M})$ denotes an unspecified function of the matrix variable $\mathbf{M}$, covariant under the action of $\mathcal{U}_q(A_N)$ described in [7]. The action (2.13) is to be evaluated on representations of the kind (2.11).

Equipped with these result, one can now try to perform constructions of branes in simple current WZW orbifold models, e.g. on the manifold $SU(N+1)/\mathbb{Z}_{N+1}$.

## 3. The quantum orbifold.

The present section is central to our paper. It contains an explicit construction of the orbifold. We also demonstrate the appearance of new quantum geometries. The latter

[4] $\eta^* R^L_\Lambda = R^{L+1}_\Lambda$ where $R^0_\Lambda \equiv R^{N+1}_\Lambda$ and where we have introduced the action $\eta^*$ on weights induced by (2.5) in an obvious manner.

[5] Cp [7].

describe fixed point D-branes of the simple current $\mathbb{Z}_{N+1}$-orbifold of the WZW model of type $A_N$.

In orbifolding, the Casimir $\mathcal{K}$ proves to play a prominent rôle. It was used in (2.7) to implement the appropriate action of $Z_{N+1}$ on the coordinate variables $M_{ij}$. Upon restricting to the set of corresponding $\mathcal{K}$-invariant monomials within $\mathrm{REA}_q(A_N)$ we recover the orbifold algebra in the form[6]:

$$\mathrm{REA}_q(A_N)/\mathbb{Z}_{N+1} = \mathrm{span}\left\langle M_{i_1 j_1} M_{i_2 j_2} \cdots M_{i_{N+1} j_{N+1}} \right\rangle / I_{RE,\mathrm{det}_q}, \tag{3.1}$$

where $I_{RE,\mathrm{det}_q}$ is the ideal generated by (2.1) and (2.4), rephrased in terms of the invariant monomials. Clearly, the orbifold algebra is unital.

Determining the geometry of D-branes in the quantum-algebraic setup requires two pieces of data: the algebra of "functions" and its representation theory. Thus it is legitimate to consider - in addition to (2.7) - the action of $\mathcal{K}$ on (2.11):

$$\mathcal{K} \triangleright v_\Lambda = R_\Lambda(\mathcal{K}) v_\Lambda = e^{\frac{2\pi i}{N+1}\mathcal{C}(\Lambda)} v_\Lambda, \qquad v_\Lambda \in \mathcal{B}_\Lambda. \tag{3.2}$$

In this case, projecting onto the subset of $\mathcal{K}$-invariant representations is tantamount to imposing the condition:

$$\mathcal{C}(\Lambda) = 0 \mod N+1. \tag{3.3}$$

The latter means that the monodromy charge (A.16) of the representation considered should vanish, in accordance with the BCFT results (cp App.A).

Altogether, the Casimir $\mathcal{K}$ is seen to realise the action of $\mathbb{Z}_{N+1} \subset \mathcal{O}ut\,(\mathrm{REA}_q(A_N))$ both at the level of the algebra and that of the associated representation theory, sending us from the original geometry - as given by $\mathrm{REA}_q(A_N)$ and $\mathcal{R}ep_{\mathrm{ind.}}(\mathrm{REA}_q(A_N))$ - to the orbifold one upon dividing out its left action on the two components.

In order to be able to assess the structure encoded in (3.1) we distinguish within the set of $\mathbb{Z}_{N+1}$-invariants the following subsets:

$$\{M^{N+1}\}\ ,\ \{\mathfrak{c}_1 M^N\}, \ldots\ \{\mathfrak{c}_N M_{ij}\ ,\ \mathfrak{c}_{N-1}\mathfrak{c}_1 M_{ij}\ ,\ \mathfrak{c}_{N-2}\mathfrak{c}_2 M_{ij}\ ,\ \ldots\ ,\ \mathfrak{c}_1^N M_{ij}\}\ ,\ \{\mathrm{det}_q \mathbf{M} \sim \mathbb{I}\}, \tag{3.4}$$

where $M^p$ symbolically denotes an arbitrary monomial in $M_{ij}'s$ of degree $p$ as in (3.1). The above decomposition shows that the orbifold geometry associated to a generic weight (on which some $\mathfrak{c}_k \neq 0$) is as in the pre-orbifolding case i.e. it is generated by $M_{ij}$'s (modulo $I_{RE,\mathrm{det}_q}$, as usual). For these weights, we may take the definition of the brane on the orbifold to be as in (2.11), supplemented by the constraint (3.3). The fixed point geometries, on the other hand, display an altogether different structure, analysed at some length in the next section.

It ought to be remarked at this point that the orbifolding described cuts the range of admissible affine weight labels $\Lambda$ to $P_+^\kappa(A_N)/\mathbb{Z}_{N+1}$, with the division determined by the action of $\omega_N^*$ (equivalent to $\eta^*$ in this case) together with the monodromy projection (3.3).

[6] The discussion below is confined to the maximal $\mathbb{Z}_{N+1}$-orbifold for the sake of concreteness only but the conclusions drawn are of general validity.

The resulting set spans an $N$-dimensional solid within $P^{\kappa}_{+}(A_N)$ with vertices defined by the vectors:

$$\mathcal{D} \;:\; \left\langle 0, \frac{\kappa}{2}\Lambda^i, \frac{\kappa}{3}\left(\Lambda^j + \Lambda^{j+1}\right), \frac{\kappa}{3}\left(\Lambda^1 + \Lambda^N\right), \frac{\kappa}{N+1}\sum_{k=1}^{N}\Lambda^k \right\rangle^{i\in\overline{1,N},\; j\in\overline{1,N-1}} . \tag{3.5}$$

One should note, in particular, the presence of the central weight $\frac{\kappa}{N+1}[1,1,\ldots,1]$ (Dynkin label notation) in (3.5). It is an actual element of $P^{\kappa}_{+}(A_N)$ whenever $N+1 \mid \kappa$ ($N+1$ divides $\kappa$) and an actual element of $P^{\kappa}_{+}(A_N)/\mathbb{Z}_{N+1}$ iff $N+1 \mid \frac{N\kappa}{2}$.

### 3.1 New geometries in $\mathbf{REA}_q(A_N)/\mathbb{Z}_{N+1}$.

The non-generic features of orbifold geometries come to the fore whenever there is a fixed point of the action of $(\omega^*_N)^D$ on $P^{\kappa}_{+}(A_N)$ for some $1 \le D < N+1$. In this case some of $\mathfrak{c}_k$ vanish[7], as follows from (2.10): since the $N+1$ independent Casimir eigenvalues pick up the $N+1$ independent phase factors under the simple current rotation $\omega^*_N$ the corresponding weight $\Lambda_{FP}$ will be stabilised by $(\omega^*_N)^D$ iff the Casimirs whose eigenvalues do change under the latter vanish. The same conclusion holds for $\eta$ due to (2.12). Among the nontrivially stabilised weights there migth be a distinguished one, sitting in the geometric centre of the $N$-simplex of the fundamental affine alcove (cp (3.5) and subsequent remarks).

We want to gain some insight into the structure of fixed point algebras. To these ends consider a restriction of the algebra of $\mathbb{Z}_{N+1}$-invariants to the representation (2.11) of $\mathrm{REA}_q(A_N)$ associated with a given weight $\Lambda$ and characterised by the following set of non-zero Casimir eigenvalues:

$$C_{k_1}, C_{k_2}, \ldots, C_{k_K} \neq 0 \tag{3.6}$$

where $C_j := \mathfrak{c}_j|_{\mathcal{B}_\Lambda},\ j \in \overline{1,N}$ and $C_i = 0$ for all $i \notin \{k_1, k_2, \ldots, k_K\}$.

**We then claim that the corresponding coordinate subalgebra is generated by independent monomials of degree $n$ such that**

$$n = \gcd\left(N+1, k_j\right)_{j\in\overline{1,K}} . \tag{3.7}$$

Here is the proof of our claim. Let $n$ be as in (3.7). Given (3.6), one can form a polynomial of degree $N+1$ (a $\mathbb{Z}_{N+1}$-invariant) $\prod_{j=1}^{K}\mathfrak{c}_{k_j}^{p_j}M^p$, where $M^p$ is an arbitrary monomial of degree $p$ and $n|p$. Thus one can effectively use the monomials $M^p$ to generate the orbifold algebra. Next one takes products $(M^p)^l$ with $l>0$ the smallest number such that $(M^p)^l = M^{p_1}\mathrm{det}_q\mathbf{M}$ for some non-zero $p_1$. Then $p_1 < p$ and $p_1$ is divisible by $n$. Continuing the procedure one concludes, after a finite number $S$ of steps, that it is monomials $M^{p_S}$ that generate the algebra, with $p_S \mid N+1$. One can subsequently perform analogous reduction with respect to each of the non-vanishing Casimirs, whereby one finally reaches the desired conclusion (3.7).

The monomials $M^n$ are invariant under $\mathbb{Z}_n \subset \mathbb{Z}_{N+1}$ generated by $\eta^D$. The subgroup $\mathbb{Z}_n$ is the stabiliser of $\Lambda$ within $\mathbb{Z}_{N+1}$. In the distinguished case of the central weight (and exclusively in that case) all $\mathfrak{c}_k = 0$ and we recover the full $\mathbb{Z}_{N+1}$ as the stabiliser, with the

[7] Recall that we have imposed $\mathcal{K} \stackrel{!}{=} 1$.

algebra of monomials of degree $N+1$ as the generating one. We expect the emergence of essentially new quantum geometries whenever the number (3.7) is greater than one. These quotient geometries are described by proper subalgebras of $\mathrm{REA}_q(A_N)$ and - as argued below - have a reduced representation theory. We plan to dwell on this subject in a separate publication.

### 3.2 The fixed point resolution.

Here we shall deal with one of the "fixed point" weights $\Lambda = \Lambda_{FP}$ which defines the group $Z_n$ as above. Following the definition (2.11) we consider

$$\bigoplus_{L=0}^{N} R^{L}_{(\omega_N^*)^L\Lambda} = \bigoplus_{x=0}^{n-1}\bigoplus_{L=0}^{D-1} R^{L+xD}_{(\omega_N^*)^L\Lambda} \tag{3.8}$$

where the equality holds due to $(\omega^*)^D\Lambda = \Lambda$ $(nD = N+1)$. The space of functions on the "fixed point" brane is generated by the monomials $M^n$. Recalling that $R^{L+lD}_{\Lambda} = R^0_{\Lambda} \otimes e^{\frac{2\pi i L}{N+1} + \frac{2\pi i l}{n}}$ (see App.C.3 and App.D.2) we thus obtain, symbolically, $\left(R^{L+lD}_{\Lambda}\right)^n = \left(R^{L}_{\Lambda}\right)^n$. From this point of view the $x$-dependence in (3.8) drops out completely on passing to the orbifold geometry. Consequently, we could choose

$$b_\Lambda = \bigoplus_{L=0}^{D-1} R^{L}_{(\omega_N^*)^L\Lambda} \tag{3.9}$$

as a possible definition of the (fractional) brane. The latter satisfies $(\eta^* b_\Lambda)^n = \left(b_{\omega_N^*\Lambda}\right)^n$ (see (2.12)), which is necessary for the geometrical meaning of $\eta$.

On the other hand, the BCFT results indicate that for each such a fixed point weight there should be $n$ different branes. This means that we need to distinguish different $x$ labels somehow. We shall do it by introducing a certain cross product extension of the algebra generated by $M^n$. Given the action of the $Z_n$-generator

$$(\eta^*)^D \left(\bigoplus_{L=0}^{D-1} R^{L+xD}_{(\omega_N^*)^L\Lambda}\right) = \bigoplus_{L=0}^{D-1} R^{L+(x+1)D}_{(\omega_N^*)^L\Lambda}, \tag{3.10}$$

we see that the original module (3.8) splits into $(\eta^*)^D$-eigenspaces as

$$\bigoplus_{L=0}^{N} R^{L}_{(\omega_N^*)^L\Lambda} = \bigoplus_{y=0}^{n-1} P_y \left(\bigoplus_{L=0}^{D-1}\bigoplus_{x=0}^{n-1} R^{L+xD}_{(\omega_N^*)^L\Lambda}\right), \tag{3.11}$$

where the projectors $P_y$ onto respective eigenspaces are

$$P_y := \frac{1}{n}\sum_{m=0}^{n-1} e^{-\frac{2\pi i y m}{n}} (\eta^*)^{D\cdot m}, \qquad P_y P_z = \delta_{y,z} P_y, \quad y,z \in \overline{0, n-1}. \tag{3.12}$$

We can then identify, for arbitrary $y \in \overline{0, n-1}$, the $\mathcal{B}^y_\Lambda$–brane:

$$\mathcal{B}^y_\Lambda := P_y \left(\bigoplus_{L=0}^{D-1}\bigoplus_{x=0}^{n-1} R^{L+xD}_{(\omega_N^*)^L\Lambda}\right). \tag{3.13}$$

The crucial property of $\mathcal{B}_\Lambda^y$ is $(\eta^*)\mathcal{B}_\Lambda^y = \mathcal{B}_{\omega_N^*\Lambda}^y$ from which $(\eta^*)^D\mathcal{B}_\Lambda^y = \mathcal{B}_\Lambda^y$ follows. We may equivalently rewrite (3.13) as

$$\mathcal{B}_\Lambda^y = b_\Lambda \otimes p_y, \tag{3.14}$$

with $p_y$ - the $y$–th irreducible, one-dimensional representation of $\mathbb{Z}_n$. Consequently, we define the space of functions on the $y$–th brane ($y = 0, 1, ..., n-1$) to be generated by

$$M^n \otimes P_y(\mathcal{K}), \tag{3.15}$$

where $P_y(\mathcal{K})$ is as in (3.12) but with $\eta^*$ replaced by $\mathcal{K}$, and $p_y\left(\mathcal{K}^D\right) = e^{\frac{2\pi i y}{n}}$. The crossed product structure is completed by defining the product of the generators (3.15). The latter is inherited from the following[8] crossed product $(M_{ij} \otimes \mathcal{K}^{D\cdot m}) \cdot (M_{i'j'} \otimes \mathcal{K}^{D\cdot m'}) = M_{ij}(\mathcal{K}^{D\cdot m} \triangleright_L M_{i'j'}) \otimes \mathcal{K}^{D\cdot(m+m')}$ and reads (symbolically)

$$(M_1^n \otimes P_x(\mathcal{K})) \cdot (M_2^n \otimes P_y(\mathcal{K})) = \delta_{x,y}\ M_1^n M_2^n \otimes P_x(\mathcal{K}). \tag{3.16}$$

The algebra thus defined, together with its representation theory, describes the so-called fractional orbifold branes, discussed in the present context e.g. in [14]. Their existence was inferred, in particular, from the structure of the fixed point boundary OPE algebra, encoding the crossed product extension of the relevant algebra of functions, as argued in [14]. Finally, let us add that the emergence of the crossed product extension of the algebra of functions at fixed points of the orbifold action is a general feature of matrix models of orbifold geometries, reflecting the existence of fractional branes ([15]).

### 3.3 Orbifold brane tensions.

An important and nontrivial test of the construction presented above is the computation of tensions of the orbifold branes within the framework developed. In so doing we follow the scheme advanced in the original papers, [7], that is - we compute the tension of a brane labelled by the weight $[\Lambda] \in \mathcal{D} \subset P_+^\kappa(A_N)$ by evaluating the first term of the effective action (2.13) on the associated representation, $\mathcal{B}_{[\Lambda_{FP}]}^y$, for a fixed point weight $\Lambda = \Lambda_{FP}$, or $\mathcal{B}_\Lambda =: \mathcal{B}_{[\Lambda]}^{off}$ for an off-fixed-point one (clearly, the orbifolding of off-fixed-point modules produces modules isomorphic to the original (pre-orbifolding) ones). Beside giving rise to the aforementioned crossed product extension of the fixed point algebra of $\mathbb{Z}_{N+1}$-invariants, the latter yields the correct result for the tension of resolved fixed point branes $\mathcal{B}_{[\Lambda_{FP}]}^y$:

$$\frac{T_0}{N+1}\mathrm{Tr}_q\big|_{\mathcal{B}_{[\Lambda_{FP}]}^y}\mathbb{I} = \frac{1}{N+1}\cdot D\cdot \mathrm{Tr}_q\big|_{R_{\Lambda_{FP}}^0}\mathbb{I} = \frac{1}{n}\mathcal{E}_{\Lambda_{FP}}, \qquad y \in \mathbb{Z}_n, \quad n = |\mathcal{S}(\Lambda_{FP})|, \tag{3.17}$$

falling in perfect agreement with the BCFT data (cp (A.7)). It is supplemented by an equally satisfactory result for generic (off-fixed-point) branes $\mathcal{B}_{[\Lambda]}^{off}$:

$$\frac{T_0}{N+1}\mathrm{Tr}_q\big|_{\mathcal{B}_{[\Lambda]}^{off}}\mathbb{I} = \frac{T_0}{N+1}\cdot(N+1)\cdot\mathrm{Tr}_q\big|_{R_\Lambda^0}\mathbb{I} = \mathcal{E}_\Lambda, \tag{3.18}$$

an immediate consequence of the lack of an internal mechanism of spectrum reduction in this last case.

[8] Cp [13].

## 4. An example: branes on $\mathbb{R}P^3_q$.

Below we detail the particularly simple example of the antipodal $\mathbb{Z}_2$-orbifold of the quantum matrix model for $\mathfrak{su}_\kappa(2)$, $\kappa \in 4\mathbb{N}^*$. The model is exemplary in that it develops precisely along the lines discussed in the previous sections, hence we restrict ourselves here to its q-geometric interpretation, relating our results to some well-established mathematical constructs of [16] and [17]. The BCFT properties of the corresponding branes were discussed in [14, 18].

### 4.1 The setup.

Our starting point shall be the RE algebra suggested in [7] as giving a plausible compact description of the quantum D-brane geometry on WZW group manifolds. Since we aim at describing the stringy geometry of the antipodal orbifold of $SU(2)$ we will focus on the RE algebra generated by the operator entries of a matrix $\mathbf{M}$ subject to the reflection equation (2.1) in which $\mathbf{R}$ is the standard (Cp [13]) universal $\mathcal{R}$-matrix of $\mathcal{U}_q(\mathfrak{su}(2))$ in the bi-fundamental representation,

$$\mathbf{R} = \sum_{i,j,k,l=1}^{2} R^{ij}_{kl} \mathbf{e}_{ik} \otimes \mathbf{e}_{jl} = q\mathbf{e}_{11} \otimes \mathbf{e}_{11} + q\mathbf{e}_{22} \otimes \mathbf{e}_{22} + \mathbf{e}_{11} \otimes \mathbf{e}_{22} + \mathbf{e}_{22} \otimes \mathbf{e}_{11} + \lambda \mathbf{e}_{12} \otimes \mathbf{e}_{21}. \quad (4.1)$$

Having parameterised $\mathbf{M}$ as

$$\mathbf{M} = \begin{pmatrix} M_4 - iq^{-2}M_0 & iq^{-\frac{1}{2}}\sqrt{[2]_q}M_{-1} \\ -iq^{-\frac{3}{2}}\sqrt{[2]_q}M_1 & M_4 + iM_0 \end{pmatrix}, \quad (4.2)$$

we may write down the additional Casimir constraint:

$$r^2\mathbb{I} \equiv \det_q \mathbf{M} = M_4^2 + M_0^2 - q^{-1}M_1M_{-1} - qM_{-1}M_1, \quad (4.3)$$

in a natural way, with $r^2$ interpreted as the radius squared of the group manifold, $\mathbb{S}^3_q$, which we set roughly proportional to the level of the underlying WZW model (cf [7]),

$$r \approx \sqrt{\alpha' \kappa_1}. \quad (4.4)$$

The RE (2.1) now takes a manageable (component) form ($i \in \{0, 1, -1\}$):

$$[M_4, M_i] = 0 \quad , \quad M_0M_{-1} = qM_{-1}(qM_0 - i\lambda M_4),$$

$$M_1M_0 = q(qM_0 - i\lambda M_4)M_1 \quad , \quad M_1M_{-1} - M_{-1}M_1 = \lambda M_0(M_0 - iM_4). \quad (4.5)$$

The above algebra, further constrained by (4.3), is easily seen to reproduce (after diagonalising the central element $M_4$ and some trivial rescalings) the celebrated Podleś' spheres, $\mathbb{S}^2_{q,c}$, with the parameter $c$ essentially determined by the value of $M_4$ (cf [16]).

Bearing in mind the clear geometric picture of the REA, we should expect that the general q-automorphism (2.5) corresponds to the standard antipodal identification of "points" on $\mathbb{S}^3_q$. In order to verify that we consider the embedding (D.2) and compute

$$\mathcal{K}^L \triangleright \mathbf{M} = re^{-\pi iL} \begin{pmatrix} q^H + q^{-1}\lambda^2 FE & q^{-1}\lambda F \\ \lambda q^{-H}E & q^{-H} \end{pmatrix}, \qquad L \in \{0, 1\}, \quad (4.6)$$

where now

$$\mathcal{K} = e^{-\pi i H}. \tag{4.7}$$

The Casimir yields

$$\mathcal{K}^L \triangleright \mathfrak{c}_1 = \mathcal{K}^L \triangleright \frac{1}{[2]_q}\mathrm{tr}_q(\mathbf{M}) = \mathcal{K}^L \triangleright M^4 \xrightarrow{\mathcal{B}_\Lambda} r e^{-\pi i L} \frac{\cos\frac{(\Lambda+1)\pi}{\kappa_1}}{\cos\frac{\pi}{\kappa_1}}\mathbb{I}. \tag{4.8}$$

Thus the quantum $\mathbb{Z}_2$-automorphism (2.5) reads

$$\eta : M_\mu \longrightarrow -M_\mu, \quad \mu \in \{4, 0, 1, -1\}, \tag{4.9}$$

just as required for the automorphism to have the interpretation of an antipodal map on $\mathbb{S}^3_q$. The monodromy operator (4.7) acts trivially only on integer-spinned irreducible representations.

We are now in a position to explicitly construct the q-matrix model for the $\mathbb{Z}_2^{antipod.}$-orbifold of the $\mathfrak{su}_\kappa(2)$ WZW model at $\kappa \in 4\mathbb{N}^*$.

## 4.2 The orbifold.

In the light of the general results of Sect.3.2. the coordinate algebra of the $\mathbb{Z}_2^{antipod.}$-orbifold of the stringy manifold $\mathbb{S}^3_q$ is the algebra of quadratic monomials in the generators $M_\mu$ of $\mathrm{REA}_q(\mathfrak{su}(2))$, $\mathbb{Z}_2$-extended - in the case of $\kappa \in 4\mathbb{N}^*$ - at the unique central fixed point, $\Lambda_{FP} = \frac{\kappa}{2}$. The algebra of $\mathbb{Z}_2$-invariants, defining a quantum manifold which could be called the real quantum projective 3-plane, $\mathbb{R}P^3_q$, is easily verified to be generated by the mutually independent operators:

$$\mathcal{O}\left(\mathbb{R}P^3_q\right) = \mathrm{span}\left\langle \mathbb{I}, M_0^2, M_1^2, M_{-1}^2, M_0M_1, M_{-1}M_0, M_4^2, M_4M_0, M_4M_1, M_4M_{-1}\right\rangle / I_{RE,\det_q}, \tag{4.10}$$

where $I_{RE,\det_q}$ is the ideal defined by a set of relations, deriving directly from (4.3) and (4.5). The relations are not very illuminating and shall therefore be left out.

An important feature of the ensuing algebra is the central character of $M_4^2$ which shall consequently be used to label inequivalent irreducible representations. Thus for $M_4^2 \neq 0$ (or $\Lambda \neq \frac{\kappa}{2}$) we recover quantum 2-spheres, as discussed in Sect.3.1 and indicated by the BCFT. At the fixed point, on the other hand, where the original RE and (4.3) simplify,

$$M_0M_{-1} = q^2 M_{-1}M_0 \quad , \quad M_1M_0 = q^2 M_0M_1,$$

$$M_1M_{-1} - M_{-1}M_1 = \lambda M_0^2 \quad , \quad M_0^2 - q^{-1}M_1M_{-1} - qM_{-1}M_1 \stackrel{!}{=} r^2\mathbb{I}, \tag{4.11}$$

the orbifold algebra reduces - upon choosing the rescaled generators:

$$(P, R, \bar{R}, T, \bar{T}) := \frac{[2]_q}{q^2r^2}\left(\frac{1}{[2]_q}M_0^2, q^2M_1^2, q^2M_{-1}^2, \frac{q}{\sqrt{[2]_q}}M_0M_1, -\frac{q}{\sqrt{[2]_q}}M_{-1}M_0\right), \tag{4.12}$$

to the $\mathbb{R}P^2_q$ algebra found in [17], generalized here to $q$ a phase and no longer real. According to the general discussion of Sec.3.2 we have two inequivalent $\mathbb{R}P^2_q$-branes corresponding to

the two inequivalent irreducible representations of the stabiliser group $\mathbb{Z}_2$ (with $D = 1$ of Sec.3.2 here).

We remark at this point that - as follows from the restriction to irreducible representations of vanishing monodromy (i.e. - in the case at hand - integer spin), effected by (3.2) - it is only the integer spin branes that compose $RP_q^3$.

**Space of functions**

Here we perform more scrupulous a comparison of the two descriptions of the fixed point geometry: the BCFT and the algebraic one. The modified (according to App.A) fixed point BCFT brane geometry decomposes as ($\kappa \in 4\mathbb{N}^*$):

$$\text{span}\left\langle \psi_{I,i;0}^{[\Lambda_{FP}]\pm[\Lambda_{FP}]\pm} \right\rangle_{I\in\overline{0,\frac{\kappa}{2}}\cap 2\mathbb{N},\ i\in\overline{0,2I}} \equiv \bigoplus_{\frac{K}{2}=0}^{\frac{\kappa}{4}} (2K+1). \tag{4.13}$$

In the matrix model, on the other hand, we are free to take arbitrary monomials in the generators $P, R, \bar{R}, T, \bar{T}$, with the (even integer) spin of any such monomial determined by its overall degree in the original coordinates $M_j,\ j \in \{0, 1, -1\}$. Restricting to the basis monomials ([17]), ordered according to their spin $2s \in 2\mathbb{N}$,

$$P^m R^{s-m},\ P^n \bar{R}^{s-n},\ P^l T R^{s-n-1},\ P^n \bar{T} \bar{R}^{s-n-1}, \qquad m \in \overline{0,s},\ n \in \overline{0,s-1}, \tag{4.14}$$

we should obtain as many independent operators as there are even-spinned primaries, which - clearly - is not the case[9] unless we truncate the spin of the monomial generators of the equatorial $\mathbb{R}P_q^2$ algebra as $s \leq \frac{\kappa}{4}$. The dimension of the space of monomials reduces accordingly,

$$\sum_{s=0}^{S>\frac{\kappa}{4}} (4s+1) \longrightarrow \sum_{s=0}^{\frac{\kappa}{4}} (4s+1) = \frac{1}{2}\left(\frac{\kappa}{2}+1\right)\left(\frac{\kappa}{2}+2\right), \tag{4.15}$$

to yield precisely the BCFT-dictated number of geometric degrees of freedom on the side of the matrix model. With this last observation we conclude our exposition of the simple example of quantum orbifold geometry.

## 5. Summary.

In the present paper, we advanced a quantum-algebraic model for curved non-commutative D-brane geometries on simple current orbifolds of the $SU(N+1)$ WZW manifolds. Following the original ideas of [7] we were able to consistently encode the BCFT data, regarded here as the ultimate foundation of all our constructions, in a simple algebraic framework based on the central idea of quantum group symmetry, as suggested

[9]The origin of the mismatch traces back to the failure of the original Drinfel'd twisting procedure to render the ensuing $q$-deformed OPE algebra associative in the entire range of spin parameters of the theory. It turns out that - as was remarked already in [7] - the associativity breaks down as early as $\Lambda = E\left(\frac{\kappa}{4}\right)$, half-way between the north pole of $\mathbb{S}_q^3$ and its equator.

by the structure of the CFT itself. Our study of the relevant Reflection Equation Algebras revealed significant structural similarities between the latter and the BCFT of the corresponding WZW models. The similarities, present both at the level of the respective representation theories ($\mathcal{O}ut\left(A_N^{(1)}\right)$ versus $\mathcal{O}ut\,(\mathrm{REA}_q(A_N))$) and at the level of the algebra (the (resolved) orbifold BA versus the (cross-product-extended) $\mathrm{REA}_q(A_N)/\mathbb{Z}_{N+1}$), provide nontrivial evidence for a close relationship between the $\mathcal{U}_q(A_N)$-related algebras $\mathrm{REA}_q(A_N)$ and the stringy geometries defined by the BCFT. An important feature of this relationship is its naturalness. It follows - in particular - from the clear geometric meaning assigned to $\mathrm{REA}_q(A_N)$, which - in turn - induces a simple realisation of classical-type symmetries (whether continuous or discrete as in the present context) of the non-commutative manifolds defined by $\mathrm{REA}_q(A_N)$. One of the important manifestations of the relationship is a straightforward identification of the CFT monodromy charge in the quantum-algebraic setup. Last but not least, the explicit physical results on tensions of orbifold branes, falling in perfect agreement with the BCFT data, lend further support to our choice of the algebraic structure deforming the underlying BCFT[10].

On the more formal side, we draw the reader's attention to the attractive pattern in the representation theory of $\mathrm{REA}_q(A_N)$ uncovered in this paper, admitting a straightforward explanation in reference to the purely geometrical symmetries of the associated affine structure $A_N^{(1)}$. An immediate consequence of its presence is the construction of an entire class of new quantum geometries, wrapped by the fractional branes of the matrix model.

Altogether, the arguments in favour of the proposal (2.11) and (2.13) and the associated orbifolding scheme are purely formal as well as physical in nature, turning our description of both the pre- and post-orbifolding $q$-geometry into a viable candidate for a quantum matrix model of (simple current orbifold) WZW geometry.

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[10]It is perhaps worth mentioning that - in addition to the calculations carried out explicitly in the paper - we are able within the present framework to test the quantum stability of the brane configurations (2.11) as well as examine the ensuing inter-brane excitations using the techniques developed in [19]. So far our results seem to indicate towards stability against decay and reproduce an asymptotically correct picture of the lightest open strings stretched between distant branes.

## Appendices.

## A. The BCFT and its quantum deformation.

### A.1 The orbifold OPE algebra.

The present paper focuses on the study of a class of quantum algebras and associated matrix models with the aim of encoding in them the physical content of stringy WZW orbifolds. It thus seems natural to begin our discussion with an exposition of some elements of the BCFT of simple current orbifolds relevant to the subsequent quantum algebraic analysis.

The orbifolding procedure for WZW models was laid out in [14], which we follow closely in this preparatory part, and leads to a formulation of string theory on quotient spaces $G/\Gamma$, with $\Gamma$ - a subgroup of the discrete group $s\mathcal{O}ut$ of strictly affine[11] automorphisms of $\hat{\mathfrak{g}}_\kappa$. It consists in dividing out the action of $\Gamma$ which - at the level of the relevant OPE - is generated by simple currents, i.e. primaries with simple fusion rules with all other primaries. Denoting by $\Lambda_g$ the weight label of a simple current corresponding to $g \in \Gamma$, we have a fusion rule:

$$\exists!_{g\Lambda \in P^\kappa_+(\mathfrak{g})} \ : \ \Lambda_g \times_{\mathcal{F}} \Lambda = g\Lambda \tag{A.1}$$

for an arbitrary weight label $\Lambda \in P^\kappa_+(\mathfrak{g})$. The Abelian group formed by simple currents under fusion is known to be isomorphic with the group of strictly affine automorphisms.

The action of $\Gamma$ on the set of all primaries, labelled by weights $\Lambda \in P^\kappa_+(\mathfrak{g})$, decomposes the latter into orbits and we take $[\Lambda]$ to label all boundary conditions associated with $\Lambda$ by that action. Some of the orbits may have fixed points, i.e. there may be weights stabilised by subgroups of $\Gamma$ for which we reserve the symbol $\mathcal{S}_\Lambda \subseteq \Gamma$. Among these there is a distinguished class of maximally stabilised weights with $\mathcal{S}_\Lambda = \Gamma$. Specialising to the case $\mathfrak{g} = A_N$ and $\Gamma = s\mathcal{O}ut$ we shall call the corresponding fixed points central. An important property of the stabiliser subgroups $\mathcal{S}_\Lambda$, used in the general BCFT construction, is their independence of the choice of a particular representative of the orbit $[\Lambda]$.

Another ingredient in the orbifolding recipe is the simple current charge $\hat{Q}_g(L) \in \mathbb{R}/2\mathbb{Z}$ of a given primary field $\Psi^{\Lambda_1\Lambda_2}_{L,i}$ with respect to the simple current corresponding to $g$, determined by the so-called braiding matrix[12] of the underlying CFT,

$$(-1)^{\hat{Q}_g(L)} := B^{(+)}_{L\Lambda_g}\begin{bmatrix} L & \Lambda_g \\ 0 & g\Lambda \end{bmatrix}. \tag{A.2}$$

A closely related object is the monodromy charge:

$$Q_g(L) := \hat{Q}_g(L) \mod 1 \quad \Longrightarrow \quad Q_g(L) = h_L + h_{\Lambda_g} - h_{gL} \mod 1, \tag{A.3}$$

[11] We give the name to the outer automorphisms of $\hat{\mathfrak{g}}_\kappa$ which are not automorphisms of the horizontal algebra $\mathfrak{g}$.

[12] Cp [20].

constant on simple current orbits for $h_{\Lambda_g} \in \mathbb{Z}$, in which case its vanishing on $[\Lambda]$ for all $g \in \Gamma$ places the orbit among those to survive the $\Gamma$-orbifolding ([21]).

The last piece of the BCFT machinery we need to deal with orbifold D-branes, in particular the fractional ones ([22]), is a little group theory of the stabilisers. Indeed, according to the general theory we should have a unique D-brane species over $\Lambda$ for any of the inequivalent one-dimensional irreducible representations of $\mathcal{S}_{[\Lambda]}$. We thus label the boundary states of the orbifold theory with the corresponding characters $e_a : \mathcal{S}_{[\Lambda]} \to U(1)$. Upon introducing the numbers:

$$d_b^{a\ L} := \frac{1}{|\mathcal{S}_{[\Lambda_1]} \cap \mathcal{S}_{[\Lambda_2]}|} \sum_{h \in \mathcal{S}_{[\Lambda_1]} \cap \mathcal{S}_{[\Lambda_2]}} e_a(h)(-1)^{\hat{Q}_h(L)} e_b(h^{-1}) \tag{A.4}$$

for any pair of overlapping stabilisers $\mathcal{S}_{[\Lambda_1]} \cap \mathcal{S}_{[\Lambda_2]} \neq \emptyset$ and $L$ such that there exists a non-zero fusion rule: $N_{\Lambda_1\ L}^{\ g\Lambda_2} \neq 0$ for some $g \in \Gamma$, we then obtain the partition functions for stabiliser-resolved orbifold D-branes ($\tau$ is the standard modular parameter and $\cdot$ stands for the element-wise product of groups):

$$Z^{orb}_{[\Lambda_1]_a [\Lambda_2]_b}(\tau) = \frac{1}{|\mathcal{S}_{[\Lambda_1]} \cdot \mathcal{S}_{[\Lambda_2]}|} \sum_{g \in \Gamma} \sum_{L \in P_+^{\kappa}(\mathfrak{g})} N_{\Lambda_1\ L}^{\ g\Lambda_2} d_b^{a\ L} \chi_L(\tau), \tag{A.5}$$

summing up to the partition function of orbifold orbits (or unresolved D-branes):

$$Z^{orb}_{[\Lambda_1][\Lambda_2]}(\tau) = \sum_{g \in \Gamma} \sum_{L \in P_+^{\kappa}(\mathfrak{g})} N_{\Lambda_1\ L}^{\ g\Lambda_2} \chi_L(\tau). \tag{A.6}$$

From (A.5) we now readily derive the tensions of the fractional branes[13] by specialising the formula to the case $\mathcal{S}_{\Lambda_2} = \{\mathrm{id}\}$ when it becomes

$$Z^{orb}_{[\Lambda_1]_a [\Lambda_2]}(\tau) = \frac{1}{|\mathcal{S}_{[\Lambda_1]}|} \sum_{g \in \Gamma} \sum_{L \in P_+^{\kappa}(\mathfrak{g})} N_{\Lambda_1\ L}^{\ g\Lambda_2} \chi_L(\tau) = \frac{1}{|\mathcal{S}_{[\Lambda_1]}|} Z^{orb}_{[\Lambda_1][\Lambda_2]}(\tau). \tag{A.7}$$

Hence the graviton coupling between the fractional D-brane carrying an arbitrary stabiliser label $a$ associated with $\mathcal{S}_{\Lambda_1}$ and an off-fixed-point one contributes the fraction of $\frac{1}{|\mathcal{S}_{[\Lambda_1]}|}$ to the overall graviton coupling between the (unresolved) D-branes, an intuitive result we shall demonstrate to be reproduced by the matrix model of Sect.3.

Having introduced all the relevant formal instruments we may now define an action of $\Gamma$ on the primaries of the pre-orbifolding theory:

$$g \rhd \Psi^{\Lambda_1 \Lambda_2}_{L,i}(x) := (-1)^{-\hat{Q}_g(L)} \Psi^{g\Lambda_1 g\Lambda_2}_{L,i}(x), \tag{A.8}$$

easily verified to be consistent with the boundary OPE (A.12) and its off-fixed-point version due to the following property of the fusing matrix ([14]):

$$\forall_{g \in \Gamma} \ : \ F_{gJ,k} \left[\begin{smallmatrix} i & j \\ gI & gK \end{smallmatrix}\right]^{\alpha_i,\alpha_j;\alpha_k}_{\kappa} = (-1)^{\hat{Q}_g(i)+\hat{Q}_g(j)-\hat{Q}_g(k)} F_{J,k} \left[\begin{smallmatrix} i & j \\ I & K \end{smallmatrix}\right]^{\alpha_i,\alpha_j;\alpha_k}_{\kappa}. \tag{A.9}$$

[13] Cp [18].

The definition (A.8) provides us with a possibility to average over $\Gamma$ primaries interpolating between off-fixed-point boundary states, whereby the associated orbifold primaries are obtained,

$$\Psi^{[\Lambda_1][\Lambda_2]}_{L,i;g}(x) := \sum_{g'\in\Gamma} g' \rhd \Psi^{\Lambda_1 g\Lambda_2}_{L,i}(x). \tag{A.10}$$

Supplementing the above formula with its fixed-point counterpart[14]:

$$\Psi^{[\Lambda_1]_a[\Lambda_2]_b}_{L,i;g} := \sum_{g_1\in\mathcal{S}_{\Lambda_1}}\sum_{g_2\in\mathcal{S}_{\Lambda_2}} \Psi^{[\Lambda_1][\Lambda_2]}_{L,i;g_1 g g_2^{-1}}(-1)^{-\hat{Q}_{g_1}(L)} e_a(g_1) e_b(g_2^{-1}), \tag{A.11}$$

with the simple current label $g$ in both (A.10) and (A.11) such that there is a non-zero fusion rule: $N_{\Lambda_1\ L}^{\ g\Lambda_2} \neq 0$, we may finally write down the OPE of the stabiliser-resolved boundary primaries:

$$\Psi^{[\Lambda_1]_{a_1}[\Lambda_2]_{a_2}}_{L_1,i_1;g_{12}}(x_1)\Psi^{[\Lambda_2]_{a_3}[\Lambda_3]_{a_4}}_{L_2,i_2;g_{23}}(x_2) = \delta_{a_2,a_3}\sum_{g\in\mathcal{S}_{\Lambda_2}}\sum_{L_3\in P^\kappa_+(\mathfrak{g})}\sum_{i_3=1}^{N^{\Lambda_3}_{\Lambda_1 L_3}} x_{12}^{h_1+h_2-h_3}\Psi^{[\Lambda_1]_{a_1}[\Lambda_3]_{a_4}}_{L_3,i_3;g_{123}g}(x_2)\times$$

$$\times (-1)^{-\hat{Q}_{g_{12}g}(L_2)} e_b(g) F_{g_{12}\Lambda_2,L_3}\begin{bmatrix} L_1 & L_2 \\ \Lambda_1 & g_{123}g\Lambda_3\end{bmatrix}^{i_1,i_2;i_3}_\kappa c^{L_1L_2L_3}_{i_1i_2i_3}, \tag{A.12}$$

where we have used the shorthand notation: $x_{12} := x_1 - x_2$ and $g_{123} := g_{12}g_{23}$. An analogous formula for off-fixed-point boundary states can be obtained from (A.12) by taking trivial stabiliser labels.

Prior to passing to the $q$-deformed OPE algebra we make, after [14], one more significant remark: according to [5] the OPE algebra of boundary primaries is a stringy deformation of the associative algebra of functions on the target geometry; the emergence of stabiliser resolution and the introduction of charge-weighted averages over $\Gamma$ in the above OPE has - in this spirit - been considered to reflect the existence of an algebraic structure called the crossed product extension of the algebra of functions on the orbifold, present in the known matrix models of fixed-point geometries ([15]). We shall have more to say about this issue in Sect.3.

### A.2 The monodromy projection.

There remains one more essential element of the CFT orbifolding procedure that we have not considered in detail so far, namely: restriction (in the orbifold theory) to the boundary states with a trivial monodromy charge. It is imposed, in particular, on the relevant partition function for closed strings on the group manifold $G$ as

$$Z_G(\tau) \longrightarrow Z_{G/\Gamma}(\tau) = Z^{\text{untwisted}}_{G/\Gamma}(\tau) + \sum_{k=1}^{|\Gamma|}\gamma^k \rhd Z^{\text{untwisted}}_{G/\Gamma}(\tau), \tag{A.13}$$

[14] A definition proved sensible in [14].

with the action of the single generator $\gamma$ of the subgroup $\Gamma$ of the simple current group[15] $\Gamma_{N+1} \cong \mathbb{Z}_{N+1}$ on the standard character bilinears:

$$Z_{G/\Gamma}^{\text{untwisted}}(\tau) = \sum_{\Lambda \in P_+^\kappa(\mathfrak{g}),\ Q_\gamma(\Lambda)=0} \chi_\Lambda(\tau)\chi_\Lambda^*(\tau) \tag{A.14}$$

defined as

$$\gamma \rhd \chi_\Lambda(\tau)\chi_\Lambda^*(\tau) := \chi_{\gamma\Lambda}(\tau)\chi_\Lambda^*(\tau), \tag{A.15}$$

where $\gamma\Lambda$ is the unique dominant integral affine weight assigned to $\Lambda \in P_+^\kappa(\mathfrak{g})$ by the simple current $\gamma$ and $Q_\gamma(\Lambda)$ is - as earlier - the monodromy charge of the weight $\Lambda$ with respect to the generating current $\gamma$.

The transition (A.13) is, in fact, an instance of averaging - in the spirit of [8] - with respect to the action of the orbifold group at the level of $G$. We may readily convince ourselves of the validity of that statement by making use of the crucial formula ([23]):

$$Q_{\gamma_{N+1}}(\Lambda) = \frac{\mathcal{C}(\Lambda)}{N+1} \mod 1 \tag{A.16}$$

establishing a simple relation between the monodromy charge of $\Lambda$ with respect to the generator $\gamma_{N+1}$ of the full simple current group, $\mathbb{Z}_{N+1}$, and the so-called congruence $\mathcal{C}(\Lambda)$ of the weight $\Lambda$. The latter is a function on $P^*(\mathfrak{g})/P(\mathfrak{g})$ (with $P(\mathfrak{g})$ - the root space of $\mathfrak{g}$), a space isomorphic to the centre $Z(G)$ of the group $G$, which - in turn - coincides with $\mathbb{Z}_{N+1}$ (cp [24]). Upon choosing the Chevalley basis for $\mathfrak{g}$ we may take the generator $c_{N+1}$ of $Z(G)$ such ($\nu$ is the so-called congruence vector of $\mathfrak{g}$)

$$c_{N+1} = e^{\frac{2\pi i}{N+1}\sum_{i=1}^N \nu_i H_i}, \quad \nu := (1, 2, \ldots, N) \tag{A.17}$$

that in restriction to the irreducible representation of the weight $\Lambda$ it yields[16]:

$$c_{N+1}|_{R_\Lambda} = e^{\frac{2\pi i \mathcal{C}(\Lambda)}{N+1}} = e^{2\pi i Q_{\gamma_{N+1}}(\Lambda)}. \tag{A.18}$$

The last identity enables us to rewrite (A.14) in the following manner:

$$\begin{aligned} Z_{G/\Gamma}^{\text{untwisted}}(\tau) &= \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}\sum_{\Lambda\in P_+^\kappa(\mathfrak{g})} R_\Lambda(\gamma)\chi_\Lambda(\tau)\chi_\Lambda^*(\tau) = \\ &= \frac{1}{|\Gamma|}\sum_{k=0}^{|\Gamma|-1}\sum_{\Lambda\in P_+^\kappa(\mathfrak{g})} R_\Lambda(c_{N+1}^{d\cdot k})\chi_\Lambda(\tau)\chi_\Lambda^*(\tau), \quad d := \frac{N+1}{|\Gamma|} \stackrel{!}{\in} \mathbb{N}, \end{aligned} \tag{A.19}$$

which completes our demonstration. The above averaging was employed quite explicitly in [3] to obtain modular invariants for the orbifolds $SU(2)/\mathbb{Z}_2$ and $SU(3)/\mathbb{Z}_3$.

[15] We are going to work with $SU(N+1)$ groups in the sequel, hence the particular choice of the rank of the simple current group.

[16] It ought to be emphasised that $\mathcal{C}$ is well-defined on irreducible representations, it being a function on $P^*(\mathfrak{g})/P(\mathfrak{g})$.

In conclusion, we note that (A.19) provides a clear geometric interpretation of the projection (A.14) onto the subset of irreducible representations of $\mathfrak{g}$ carrying the trivial monodromy charge, namely: it should be understood as a transition from the $(N+1)$-element set[17] of congruence classes of adjoint orbits (or conjugacy classes, labelled by the corresponding weights) within the group manifold of $G$ to the $d$-element subset consisting of all congruence classes associated with the unique non-faithful (trivial) irreducible representation of $\Gamma \subset Z(G)$. The transition, on the other hand, follows from dividing out the action of simple currents which - at the level of $G$ - become encoded in $Z(G)$.

It seems well worth remarking that the averaging over the subset $\Gamma$ of the centre $Z(G)$ of the group descends from the somewhat abstract level of partition functions to the level of the orbifold boundary OPE algebras. Indeed, a closer look at (A.10) and (A.8) reveals the presence of the familiar structure[18]:

$$\Psi^{[\Lambda][\Lambda]}_{L,i;\gamma}(x) = \sum_{\gamma' \in \Gamma \subset Z(G)} R_L(\gamma') \Psi^{\gamma'\Lambda\,\gamma'\gamma\Lambda}_{L,i}(x). \tag{A.20}$$

Given the geometric interpretation of both the projection and the boundary OPE algebras themselves it is quite natural to expect the above general pattern to repeat itself in the quantum-algebraic setup to be developed.

## B. Quantum deformation.

The first step towards an algebraic description of WZW D-branes is taken at the level of the general boundary OPE:

$$\Psi^{\Lambda_1\Lambda_2}_{L_1,i_1}(x_1)\Psi^{\Lambda_2\Lambda_3}_{L_2,i_2}(x_2) \sim_{OPE} \sum_{L_3 \in P^\kappa_+(\mathfrak{g})} \sum_{i_3=1}^{N^{\Lambda_3}_{\Lambda_1\ L_3}} x_{12}^{h_1+h_2-h_3} \Psi^{\Lambda_1\Lambda_3}_{L_3,i_3}(x_2) F_{\Lambda_2 L_3}\left[\begin{smallmatrix} L_1 & L_2 \\ \Lambda_1 & \Lambda_3 \end{smallmatrix}\right]^{i_1,i_2;i_3}_\kappa c^{L_1L_2L_3}_{i_1i_2i_3}, \tag{B.1}$$

in which $F_{\Lambda_2 L_3}\left[\begin{smallmatrix} L_1 & L_2 \\ \Lambda_1 & \Lambda_3 \end{smallmatrix}\right]^{i_1,i_2;i_3}_\kappa$ is the fusion matrix of the model and $c^{L_1L_2L_3}_{i_1i_2i_3}$ are structure constants encoding the group-theoretic nature of the indices carried by the horizontal descendants $\Psi^{\Lambda_1\Lambda_2}_{L_1,i_1}$ of boundary primary fields $\Psi^{\Lambda_1\Lambda_2}_{L_1}$ (for details cp, e.g., [1]).

With the aim of extracting from (B.1) the D-brane geometry we make the usual assignment (cp [25] and earlier papers on non-commutative stringy geometries):

$$\Psi^{\Lambda_1\Lambda_2}_{L_1,i_1}(x_1) \longrightarrow \psi^{\Lambda_1\Lambda_2}_{L_1,i_1}, \tag{B.2}$$

so that the algebraic content of (B.1) is preserved,

$$\psi^{\Lambda_1\Lambda_2}_{L_1,i_1} \star \psi^{\Lambda_2\Lambda_3}_{L_2,i_2} := \sum_{L_3 \in P^\kappa_+(\mathfrak{g})} \sum_{i_3=1}^{N^{\Lambda_3}_{\Lambda_1\ L_3}} \psi^{\Lambda_1\Lambda_3}_{L_3,i_3} F_{\Lambda_2 L_3}\left[\begin{smallmatrix} L_1 & L_2 \\ \Lambda_1 & \Lambda_3 \end{smallmatrix}\right]^{i_1,i_2;i_3}_\kappa c^{L_1L_2L_3}_{i_1i_2i_3}, \tag{B.3}$$

[17] The number $N+1$ is known in the present context as the index of connection of $G$.

[18] We restrict ourselves to the off-fixed-point case for clarity of the exposition only.

and only the world-sheet dependence is dropped[19]. The Boundary Algebra (BA) thus defined is readily demonstrated to be non-associative and so it has to be deformed to be embedded in a matrix algebra. The non-associativity follows from its hybrid quantum-classical structure. Indeed, while the three-point structure constants $c_{i_1 i_2 i_3}^{L_1 L_2 L_3}$ are classical intertwiners (in the simplest case of the $\mathfrak{su}_\kappa(2)$ model they are just the ordinary Clebsch–Gordan coefficients of $SU(2)$) the fusion matrix is already a quantum entity[20]. The last observation leads us to the idea of deforming (B.3),

$$c_{i_1 i_2 i_3}^{L_1 L_2 L_3} \longrightarrow \tilde{c}_{i_1 i_2 i_3}^{L_1 L_2 L_3}\,, \tag{B.4}$$

in such a way:

$$\sum_{l,\alpha_l} F_{J,l}\left[\begin{smallmatrix} i & j \\ I & K \end{smallmatrix}\right]_\kappa^{\alpha_i,\alpha_j;\alpha_l} F_{K,m}\left[\begin{smallmatrix} l & k \\ I & L \end{smallmatrix}\right]_\kappa^{\alpha_l,\alpha_k;\alpha_m} \tilde{c}_{ijl}^{\alpha_i\alpha_j\alpha_l}\, \tilde{c}_{lkm}^{\alpha_l\alpha_k\alpha_m} =$$

$$= \sum_{l,\alpha_l} F_{J,m}\left[\begin{smallmatrix} i & l \\ I & L \end{smallmatrix}\right]_\kappa^{\alpha_i,\alpha_l;\alpha_m} F_{K,l}\left[\begin{smallmatrix} j & k \\ J & L \end{smallmatrix}\right]_\kappa^{\alpha_j,\alpha_k;\alpha_l} \tilde{c}_{jkl}^{\alpha_j\alpha_k\alpha_l}\, \tilde{c}_{ilm}^{\alpha_i\alpha_l\alpha_m} \tag{B.5}$$

as to turn the latter purely quantum and associative[21], so that

$$\left(\psi_{L_1,i_1}^{\Lambda_1\Lambda_2} \star_q \psi_{L_2,i_2}^{\Lambda_2\Lambda_3}\right) \star_q \psi_{L_3,i_3}^{\Lambda_3\Lambda_4} = \psi_{L_1,i_1}^{\Lambda_1\Lambda_2} \star_q \left(\psi_{L_2,i_2}^{\Lambda_2\Lambda_3} \star_q \psi_{L_3,i_3}^{\Lambda_3\Lambda_4}\right). \tag{B.6}$$

obtains for $\star_q$ defined as $\star$ in (B.3) but with the substitution (B.4). In the abovementioned $\mathfrak{su}_\kappa(2)$ case the deformation boils down to replacing the classical Clebsch–Gordan coefficients with those of $\mathcal{U}_q(\mathfrak{su}(2))$ and was given in [25], where it first appeared, an interpretation in terms of the so-called Drinfel'd twist. The idea was developed and exploited in [7], with essential emphasis on the indication, contained in the new boundary algebra, towards a quantum symmetry of the underlying D-brane geometry and, consequently, of an associated matrix model, built on the assumption of quantum group covariance. Indeed, by adducing the standard Wigner–Eckhart argument, we identify the new boundary operators $\psi_{L_1,i_1}^{\Lambda_1\Lambda_2}$ with certain $\mathcal{U}_q(\mathfrak{g})$-intertwiners[22], furnishing an adjoint module of the quantum algebra. A transition from the left-right $\mathcal{U}_q(\mathfrak{g})$-symmetry of the "bulk" to its thus encoded vector part lies at the heart of the quantum-algebraic framework presented. It is in this sense that the new algebra constitutes the basis, on the BCFT side, of the models developed in [7].

There is yet another feature of (B.4) which becomes particularly significant in our present context. The deformation (B.4)-(B.5) proves sufficient to turn the orbifold

[19]One can motivate this transition by considering the $\kappa \to \infty$ limit of (B.1) for a set of boundary labels $\Lambda_i$ sufficiently close to the trivial weight.

[20]The matrix enters the BCFT analysis at a rather abstract stage and both its rôle and uniqueness - a consequence of strict algebraic constraints imposed upon it - determine it as an already quantum(-group-theoretic) object. Cp [20, 26, 27].

[21]At least within a certain (truncated) range of its group-theoretic labels, cp [7].

[22]Cp [28].

analogue of (B.3) associative. The proof of the last statement is presented below and enables us to start the construction of the matrix model of the orbifold physics directly at the level of the original quantum matrix algebra of [7] and seek for the automorphisms of the latter corresponding to the elements of the simple current orbifold group. The ensuing quotient structure is expected to define the quantum geometry of untwisted D-branes wrapping the orbifold.

### B.1 Proof of the associativity of the deformed boundary algebra.

In this appendix we explicitly prove the useful fact: given (B.6), the deformation of the orbifold Boundary Algebra,

$$\psi^{[\Lambda_1]_{a_1}[\Lambda_2]_{a_2}}_{L_1,i_1;g_1} \star_q \psi^{[\Lambda_2]_{a_3}[\Lambda_3]_{a_4}}_{L_2,i_2;g_2} =$$

$$= \delta_{a_2,a_3} \sum_{g\in\mathcal{S}_{\Lambda_2}} \sum_{L_3,i_3} \psi^{[\Lambda_1]_{a_1}[\Lambda_3]_{a_4}}_{L_3,i_3;gg_{12}} e_{a_2}(g)(-1)^{-\hat{Q}_{gg_1}(L_2)} F_{g_1\Lambda_2,L_3}\left[\begin{smallmatrix} L_1 & L_2 \\ \Lambda_1 & gg_{12}\Lambda_3 \end{smallmatrix}\right]^{i_1,i_2;i_3}_{\kappa} \tilde{c}^{L_1L_2L_3}_{i_1i_2i_3}, \tag{B.7}$$

renders the latter associative (we retrieve the unresolved case from (B.7) upon setting $g = \mathrm{id} = g'$):

$$\left(\psi^{[\Lambda_1]_{a_1}[\Lambda_2]_{a_2}}_{L_1,i_1;g_1} \star_q \psi^{[\Lambda_2]_{a_3}[\Lambda_3]_{a_4}}_{L_2,i_2;g_2}\right) \star_q \psi^{[\Lambda_3]_{a_5}[\Lambda_4]_{a_6}}_{L_3,i_3;g_3} =$$

$$= \delta_{a_2,a_3} \sum_{g\in\mathcal{S}_{\Lambda_2}} \sum_{L_4,i_4} \psi^{[\Lambda_1]_{a_1}[\Lambda_3]_{a_4}}_{L_4,i_4;gg_{12}} \star_q \psi^{[\Lambda_3]_{a_5}[\Lambda_4]_{a_6}}_{L_3,i_3;g_3} e_{a_2}(g)(-1)^{-\hat{Q}_{gg_1}(L_2)} \times$$

$$\times F_{g_1\Lambda_2,L_4}\left[\begin{smallmatrix} L_1 & L_2 \\ \Lambda_1 & gg_{12}\Lambda_3 \end{smallmatrix}\right]^{i_1,i_2;i_4}_{\kappa} \tilde{c}^{L_1L_2L_4}_{i_1i_2i_4} =$$

$$= \delta_{a_2,a_3}\delta_{a_4,a_5} \sum_{\substack{g\in\mathcal{S}_{\Lambda_2} \\ g'\in\mathcal{S}_{\Lambda_4}}} \sum_{\substack{L_4,i_4 \\ L_5,i_5}} \psi^{[\Lambda_1]_{a_1}[\Lambda_4]_{a_6}}_{L_5,i_5;g'gg_{123}} e_{a_4}(g') e_{a_2}(g)(-1)^{-\hat{Q}_{g'gg_{12}}(L_3)-\hat{Q}_{gg_1}(L_2)} \times$$

$$\times F_{gg_{12}\Lambda_3,L_5}\left[\begin{smallmatrix} L_4 & L_3 \\ \Lambda_1 & g'gg_{123}\Lambda_4 \end{smallmatrix}\right]^{i_4,i_3;i_5}_{\kappa} F_{g_1\Lambda_2,L_4}\left[\begin{smallmatrix} L_1 & L_2 \\ \Lambda_1 & gg_{12}\Lambda_3 \end{smallmatrix}\right]^{i_1,i_2;i_4}_{\kappa} \tilde{c}^{L_1L_2L_4}_{i_1i_2i_4} \tilde{c}^{L_4L_3L_5}_{i_4i_3i_5} =$$

$$= \delta_{a_2,a_3}\delta_{a_4,a_5} \sum_{\substack{g\in\mathcal{S}_{\Lambda_2} \\ g'\in\mathcal{S}_{\Lambda_3}}} \sum_{\substack{L_4,i_4 \\ L_5,i_5}} \psi^{[\Lambda_1]_{a_1}[\Lambda_4]_{a_6}}_{L_5,i_5;g'gg_{123}} e_{a_4}(g') e_{a_2}(g)(-1)^{-\hat{Q}_{gg_1}(L_4)-\hat{Q}_{g'g_2}(L_3)} \times$$

$$\times F_{g_2\Lambda_3,L_4}\left[\begin{smallmatrix} L_2 & L_3 \\ \Lambda_2 & g'g_{23}\Lambda_4 \end{smallmatrix}\right]^{i_2,i_3;i_4}_{\kappa} F_{g_1\Lambda_2,L_5}\left[\begin{smallmatrix} L_1 & L_4 \\ \Lambda_1 & g'gg_{123}\Lambda_4 \end{smallmatrix}\right]^{i_1,i_4;i_5}_{\kappa} \tilde{c}^{L_1L_2L_4}_{i_1i_2i_4} \tilde{c}^{L_4L_3L_5}_{i_4i_3i_5} =$$

$$= \delta_{a_4,a_5} \sum_{g' \in \mathcal{S}_{\Lambda_3}} \sum_{L_4,i_4} \psi^{[\Lambda_1]_{a_1}[\Lambda_2]_{a_2}}_{L_1,i_1;g_1} \star_q \psi^{[\Lambda_2]_{a_3}[\Lambda_4]_{a_6}}_{L_4,i_4,g'g_{23}} e_{a_4}(g')(-1)^{-\hat{Q}_{g'g_2}(L_3)} \times$$

$$\times F_{g_2\Lambda_3,L_4} \left[\begin{smallmatrix} L_2 & L_3 \\ \Lambda_2 & g'g_{23}\Lambda_4 \end{smallmatrix}\right]^{i_2,i_3;i_4}_{\kappa} \tilde{c}^{L_2L_3L_4}_{i_2i_3i_4} =$$

$$= \psi^{[\Lambda_1]_{a_1}[\Lambda_2]_{a_2}}_{L_1,i_1;g_1} \star_q \left( \psi^{[\Lambda_2]_{a_3}[\Lambda_3]_{a_4}}_{L_2,i_2;g_2} \star_q \psi^{[\Lambda_3]_{a_5}[\Lambda_4]_{a_6}}_{L_3,i_3;g_3} \right) \square. \tag{B.8}$$

## C. The algebra $\mathcal{U}_{\mathbf{q}}^{\mathrm{ext}}(\mathbf{A_N})$.

We begin with the definition of $\mathcal{U}_q^{\mathrm{ext}}(A_N)$ which we take to be generated by the elements:

$$k_{\pm\epsilon_i}, \quad i \in \overline{1, N+1} \quad ; \quad E_j, F_j, \quad j \in \overline{1, N}, \tag{C.1}$$

subject to the relations [23]:

$$k_{\epsilon_i}k_{\epsilon_j} = k_{\epsilon_j}k_{\epsilon_i} \quad , \quad k_{\epsilon_i}k_{-\epsilon_i} = \mathbb{I} = k_{-\epsilon_i}k_{\epsilon_i}, \tag{C.2}$$

$$k_{\epsilon_1}k_{\epsilon_2}\cdots k_{\epsilon_{N+1}} = \mathbb{I}, \tag{C.3}$$

$$k_{\epsilon_i}E_jk_{-\epsilon_i} = q^{\delta_{ij}-\delta_{i-1,j}}E_j \quad , \quad k_{\epsilon_i}F_jk_{-\epsilon_i} = q^{-\delta_{ij}+\delta_{i-1,j}}F_j, \tag{C.4}$$

$$[E_i, F_j] = \delta_{ij}\frac{k_{\epsilon_i}k_{-\epsilon_{i+1}} - k_{-\epsilon_i}k_{\epsilon_{i+1}}}{\lambda}, \tag{C.5}$$

$$E_i^2E_j - [2]E_iE_jE_i + E_jE_i^2 = 0 \quad , \quad F_i^2F_j - [2]F_iF_jF_i + F_jF_i^2 = 0 \quad \text{for } |\mathrm{i}-\mathrm{j}| = 1, \tag{C.6}$$

$$E_iE_j = E_jE_i \quad , \quad F_iF_j = F_jF_i \quad \text{for } |\mathrm{i}-\mathrm{j}| > 1. \tag{C.7}$$

The Cartan generators $k_{\pm\epsilon_j}, j \in \overline{1, N+1}$ are defined in direct reference to the standard embedding of the root space of the classical algebra, $P(A_N)$, in $\mathbb{R}^{N+1}$. We thus have the well-known transformation between the simple-root basis, $\alpha_i$, and the orthonormal (Cartesian) one, $\epsilon_i$,

$$\alpha_i = \epsilon_i - \epsilon_{i+1}, \quad i \in \overline{1, N}, \qquad (\epsilon_i, \epsilon_j) = \delta_{ij}, \tag{C.8}$$

with the orthogonality condition on weights $\Lambda$: $\sum_{i=1}^{N+1}(\Lambda, \epsilon_i) \overset{!}{=} 0$. Clearly, the orthogonality condition is encoded in (C.3) at the level of the algebra.

The algebra is endowed with a Hopf structure. In particular, it has the antipode:

$$Sk_{\pm\epsilon_i} = k_{\pm\epsilon_i} \quad , \quad SE_i = -k_{-\epsilon_i}k_{\epsilon_{i+1}}E_i \quad , \quad SF_i = -F_ik_{\epsilon_i}k_{-\epsilon_{i+1}}, \tag{C.9}$$

employed frequently in the sequel.

[23] We denote $\lambda := q - q^{-1}$, $[2] := q + q^{-1}$ and $k_{\pm n\epsilon_i} := k^n_{\pm\epsilon_i}$, $n \in \mathbb{N}$

### C.1 The centre of $\mathcal{U}_q^{\text{ext}}(A_N)$.

Another interesting aspect of the general theory of the extended quantum enveloping algebras is the structure of their centre, $Z_q(A_N)$, playing a crucial rôle in any representation-theoretic analysis. In the case of the deformation parameter $q$ being the $2\kappa_N$-th primitive root of unity the centre is known ([12]) to be generated by the scalar operators:

$$Z_0 = \text{span}\left\langle k_{\epsilon_i}^{2\kappa_N}, e_{ij}^{2\kappa_N}, f_{ij}^{2\kappa_N}\right\rangle_{i,j\in\overline{1,N+1}}, \tag{C.10}$$

peculiar to the root-of-unity case, and the Casimir (scalar) operators:

$$Z_1 = \text{span}\left\langle \mathcal{C}_k\right\rangle_{k\in\overline{1,N}}, \tag{C.11}$$

explicitly given by (2.2) through (D.2). On the irreducible highest weight representations $e_{ij}^{2\kappa_N} = 0 = f_{ij}^{2\kappa_N}$.

### C.2 Relations between $\mathbf{REA}_q(A_N)$, $\mathcal{U}_q^{ext}(A_N)$ and $\mathcal{U}_h(A_N)$.

There is a set of algebra homomorphisms:

$$\begin{array}{c} \mathcal{U}_q(A_N) \to \mathcal{U}_q^{ext}(A_N) \to \mathcal{U}_h(A_N) \\ \uparrow \\ \text{REA}_q(A_N) \end{array} \tag{C.12}$$

which we describe in the course of the paper. We denote the corresponding generators as:

- the quantum enveloping algebra $\mathcal{U}_q(A_N)$:
  $\{K_j, K_j^{-1}, E_j, F_j\}_{j\in\overline{1,N}}$;
- the extended quantum enveloping algebra $\mathcal{U}_q^{\text{ext}}(A_N)$:
  $\{k_{\pm\epsilon_i}, E_j, F_j\}_{i\in\overline{1,N+1},\ j\in\overline{1,N}}$;
- the $h$-adic Hopf algebra $\mathcal{U}_h(A_N)$:
  $\{H_j, E_j, F_j\}_{j\in\overline{1,N}}$.

The only nontrivial images of the generators, given by the homomorphisms (C.12), are

$$\forall_{j\in\overline{1,N}} \ : \ K_j \to k_{\epsilon_j} k_{-\epsilon_{j+1}}, \tag{C.13}$$

$$\forall_{i\in\overline{1,N+1}} \ : \ k_{\epsilon_i} \to q^{H_{\epsilon_i}}, \tag{C.14}$$

where ($\Lambda_j$ are the fundamental weights)

$$H_{\epsilon_i} := \sum_{j=1}^{N} (\epsilon_i, \Lambda_j) H_j. \tag{C.15}$$

### C.3 $\mathcal{R}ep(\mathcal{U}_q^{\rm ext}(A_N))$ and $\mathcal{O}ut(\mathcal{U}_q^{\rm ext}(A_N))$.

Automorphisms of $\mathcal{U}_q^{\rm ext}(A_N)$ are of the phase-changing type[24]:

$$(k_{\pm\epsilon_i}, E_j, F_j) \longrightarrow \left(e^{\pm i\pi p_i} k_{\pm\epsilon_i}, E_j, e^{i\pi(p_j - p_{j+1})} F_j\right), \quad i \in \overline{1, N+1},\ j \in \overline{1, N}, \tag{C.16}$$

$$2(p_j - p_{j+1}) = 0 \mod 2 \quad , \quad \sum_{l=1}^{N+1} p_l = 0 \mod 2. \tag{C.17}$$

As the homomorphisms (C.12) identify the ladder generators $(E_j, F_j)$ of the algebras considered, their finite dimensional irreducible highest weight modules are isomorphic and have the same highest weight state (denoted as $V_\Lambda$). The only effect of the automorphisms (C.16)-(C.17) is a change of the phases of the eigenvalues of the Cartan generators $k_{\pm\epsilon_i}$.

We need to determine the range of $p_i's$. According to the representation theory of $\mathcal{U}_q(A_N)$ (Chapter 7 of [29]) and (C.13)-(C.14),

$$k_{\epsilon_j} k_{j+1}^{-1} \stackrel{!}{=} e^{i\pi\omega_j} q^{H_j}, \quad \omega := (\omega_1, \omega_2, \dots, \omega_N) \in \mathbb{Z}_2^N. \tag{C.18}$$

Imposing (C.3) we further constrain the parameters in (C.18):

$$\begin{aligned} \forall_{j\in\overline{2,N+1}} \ : \ k_{\epsilon_j} &\stackrel{!}{=} e^{-i\pi\sum_{m=1}^{j-1}\omega_m} q^{-\sum_{n=1}^{j-1} H_n} k_{\epsilon_1}, \\ k_{\epsilon_1}^{N+1} &\stackrel{!}{=} e^{i\pi\sum_{m=1}^{N}(N+1-m)\omega_m} q^{\sum_{n=1}^{N}(N+1-n)H_n}, \end{aligned} \tag{C.19}$$

which means that we have an $(N+1)$-tuple of automorphisms:

$$\begin{aligned} k_{\epsilon_1} &\rightarrow e^{-\frac{i\pi L(l,\omega)}{N+1}} k_{\epsilon_1}, \\ \forall_{j\in\overline{2,N+1}} \ : \ k_{\epsilon_j} &\rightarrow e^{-\frac{i\pi L(l,\omega)}{N+1}} e^{-i\pi\sum_{m=1}^{j-1}\omega_m} k_{\epsilon_j}, \end{aligned} \tag{C.20}$$

where

$$L(l, \omega) = 2l + \sum_{m=1}^{N}(N+1-m)\omega_m, \quad l \in \mathbb{Z}_{N+1}. \tag{C.21}$$

The latter provide also the most general solution to (C.17) upon a straightforward identification of phases $p_i$. The associated group $\mathcal{O}ut(\mathcal{U}_q^{\rm ext}(A_N))$ is

$$\mathcal{O}ut(\mathcal{U}_q^{\rm ext}(A_N)) = \left(\mathbb{Z}_2^N \otimes \mathbb{Z}_{N+1}\right) \ltimes \mathbb{Z}_2, \tag{C.22}$$

where the distinguished $\mathbb{Z}_2$ factor corresponds to the classical mirror symmetry of the Dynkin diagram, present already in $\mathcal{O}ut(\mathcal{U}_h(A_N))$ ([29]).

[24] We do not consider the standard Dynkin diagram reflection.

The irreducible highest weight representations $\mathcal{R}_\Lambda^{l,\omega}$ of $\mathcal{U}_q^{\rm ext}(A_N)$ are then of the form:

$$\mathcal{R}_\Lambda^{l,\omega} \subset \mathcal{R}ep_{\rm h.w.}(\mathcal{U}_q^{\rm ext}(A_N)) \cong \mathcal{R}ep_{\rm r.h.w.}(\mathcal{U}_h(A_N)) \otimes \left(\mathbb{Z}_{N+1} \otimes \mathbb{Z}_2^N\right), \tag{C.23}$$

as follows from (C.3). Beware: we do not claim that all of them are inequivalent although it is certainly true for an arbitrary weight $\Lambda \in P_+^\kappa(A_N)$ such that all Casimir operators are non-zero on, say, $\mathcal{R}_\Lambda^{0,0}$.

## D. The algebra $\mathbf{REA}_q(A_N)$.

In this paper we are interested in representations of $\mathrm{REA}_q(A_N)$ induced by the homomorphism:

$$\mathrm{REA}_q(A_N) \rightarrow \mathcal{U}_q^{ext}(A_N). \tag{D.1}$$

The homomorphism was originally discussed in [27, 30] (following the earlier results of [31]) and reads

$$M = \mathbf{L}^+ S \mathbf{L}^- \in \mathrm{Mat}((N+1)\times(N+1);\mathbb{C}) \otimes \mathcal{U}_q^{\rm ext}(A_N). \tag{D.2}$$

where

$$\mathbf{L}^\pm = \sum_{i,j=1}^{N+1} \mathbf{e}_{ij} \otimes L_{ij}^\pm \tag{D.3}$$

are operator-valued matrices[25] presented explicitly below. Notice that with (D.2) we automatically have $\det_q(M) = \mathbb{I}$ ([7]).

### D.1 $\mathbf{L}^\pm$-operators

Below we explicitly list the entries of the $\mathbf{L}^\pm$-operators obtained from the standard universal $\mathcal{R}$-matrices of $\mathcal{U}_q(A_N)$ by means of the algorithm of Faddeev, Reshetikhin and Takhtajan ([32]):

$$\forall_{i,j\in\overline{1,N+1}} \;:\; \begin{cases} L_{ii}^+ = k_{\epsilon_i} = \left(L_{ii}^-\right)^{-1}, & \\ L_{ij}^+ = 0 = L_{ji}^- & \text{for i > j,} \\ L_{ij}^+ = \lambda k_{\epsilon_i} E_{ji} & \text{for i < j,} \\ L_{ij}^- = -\lambda E_{ji} k_{-\epsilon_j} & \text{for i > j,} \end{cases} \tag{D.4}$$

[25] $\mathbf{e}_{ij}$ are the basis matrices $(\mathbf{e}_{ij})_{kl} = \delta_{ik}\delta_{jl}$.

where we have introduced the recursively defined symbols:

$$\forall_{i,j\in\overline{1,N}} \ : \ \begin{cases} E_{i,i+1} = E_i, & \\ E_{i,j+1} = E_{ij}E_j - qE_jE_{ij} & \text{for i < j}, \\ E_{i+1,i} = F_i, & \\ E_{i+1,j} = E_{i+1,j+1}F_j - q^{-1}F_jE_{i+1,j+1} & \text{for i > j}. \end{cases} \tag{D.5}$$

The above $\mathbf{L}^{\pm}$-operators are known ([32]) to satisfy the following relations:

$$\mathbf{R}^+_{12}\mathbf{L}^{\pm}_1\mathbf{L}^{\pm}_2 = \mathbf{L}^{\pm}_2\mathbf{L}^{\pm}_1\mathbf{R}^+_{12}, \tag{D.6}$$

$$\mathbf{L}^+_1\mathbf{R}^+_{12}S\mathbf{L}^-_2 = S\mathbf{L}^-_2\mathbf{R}^+_{12}\mathbf{L}^+_1, \tag{D.7}$$

and hence also

$$\mathbf{R}^+_{21}S\mathbf{L}^{\pm}_1S\mathbf{L}^{\pm}_2 = S\mathbf{L}^{\pm}_2S\mathbf{L}^{\pm}_1\mathbf{R}^+_{21}, \tag{D.8}$$

$$\mathbf{L}^+_2\mathbf{R}^+_{21}S\mathbf{L}^-_1 = S\mathbf{L}^-_1\mathbf{R}^+_{21}\mathbf{L}^+_2, \tag{D.9}$$

with

$$\mathbf{R}^+ = \sum_{1\leq i,j\leq N+1} q^{\delta_{ij}}\mathbf{e}_{ii}\otimes\mathbf{e}_{jj} + \lambda\sum_{1\leq i<j\leq N+1}\mathbf{e}_{ij}\otimes\mathbf{e}_{ji}. \tag{D.10}$$

At this stage it is a matter of an elementary algebra to verify that the fundamental $\mathbf{M}$-matrix, (D.2), with the operator entries:

$$M_{ij} = \sum_{k=1}^{N+1} L^+_{ik}SL^-_{kj}, \tag{D.11}$$

does indeed satisfy (2.1) with $\mathbf{R} = \mathbf{R}^+$.

Finally, upon substituting (D.4) in the above formula and rearranging the resulting expressions we derive

$$M_{ij}\big|_{i>j} = (-1)^{j+1-i}\lambda k_{2\epsilon_i}\left[\tilde{E}_{ji} + q^{-2}\lambda\sum_{i<k\leq N+1}(-1)^{i-k}k_{-\epsilon_i}k_{\epsilon_k}E_{ki}\tilde{E}_{jk}\right], \tag{D.12}$$

$$M_{ij}\big|_{i<j} = q^{-1}\lambda k_{\epsilon_i}k_{\epsilon_j}\left[E_{ji} + \lambda\sum_{j<k\leq N+1}(-1)^{k+1-j}k_{-\epsilon_j}k_{\epsilon_k}E_{ki}\tilde{E}_{jk}\right], \tag{D.13}$$

$$M_{ii} = k_{2\epsilon_i}\left[1 + q^{-1}\lambda^2\sum_{i<k\leq N+1}(-1)^{k+1-i}k_{-\epsilon_i}k_{\epsilon_k}E_{ki}\tilde{E}_{ik}\right], \tag{D.14}$$

with the operators $\tilde{E}_{ij}$ defined in analogy to (D.5),

$$\forall_{i,j\in\overline{1,N}} \ : \ \begin{cases} \tilde{E}_{i,i+1} = E_i, \\ \\ \tilde{E}_{i,j+1} = q^{-1} E_j \tilde{E}_{ij} - \tilde{E}_{ij} E_j \text{ for i < j}, \end{cases} \tag{D.15}$$

and further related to the $E_{ij}$'s through

$$\forall_{i,j\in\overline{1,N+1},\, i<j} \ : \ SE_{ij} = (-1)^{j-i} k_{-\epsilon_i} k_{\epsilon_j} \tilde{E}_{ij}. \tag{D.16}$$

Amongst the $(N+1)^2$ entries of the $\mathbf{M}$-matrix there is a distinguished group of the diagonal ones, $M_{ii}$, of which $N$ can be chosen to commute with one another[26] and therefore span the Cartan subalgebra of $\mathrm{REA}_q(A_N)$. We shall have need for them in the sequel.

The convention on the $\mathbf{L}^{\pm}$-operators just displayed proves exceptionally convenient for the analysis to come. Finally, let us also note that the Casimir operators of the underlying $\mathrm{REA}_q(A_N)$ translate naturally into Casimir operators of $\mathcal{U}_q^{\mathrm{ext}}(A_N)$ described by means of (D.12)-(D.14),

$$\mathfrak{c}_k \longrightarrow \mathcal{C}_k. \tag{D.17}$$

**D.2 Representations of $\mathbf{REA}_q(A_N)$.**

The irreducible highest weight representations $\mathcal{R}^L_\Lambda$ of $\mathrm{REA}_q(A_N)$ induced by (D.1) are of the form:

$$\mathcal{R}ep_{\mathrm{ind.}}(\mathrm{REA}_q(A_N)) = \bigoplus_{\Lambda\in P^\kappa_+(A_N)} \bigoplus_{L\in\mathbb{Z}_{N+1}} R^L_\Lambda, \quad R^L_\Lambda \sim R^{l,\omega}_\Lambda|_{L(l,\omega)=L \mod 2}. \tag{D.18}$$

Below we demonstrate that the representations: $R^L_\Lambda,\ (\Lambda, L) \in P^\kappa_+(A_N) \times \mathbb{Z}_{N+1}$ are indeed pairwise inequivalent.

By (2.6) and (2.10) all $R^l_{(\omega^*_N)^l\Lambda}$ for $l \in \mathbb{Z}_{N+1}$ have equal scalar operators. It is therefore the latter that we focus on in the sequel[27], further reducing our problem by making the following observation:

$$\left(\exists_{\Lambda\in P^\kappa_+(A_N)} \exists_{l_1,l_2\in\mathbb{Z}_{N+1}} : R^{l_1}_{(\omega^*_N)^{l_1}\Lambda} \sim R^{l_2}_{(\omega^*_N)^{l_2}\Lambda}\right) \Longrightarrow R^{l_2-l_1}_{(\omega^*_N)^{l_2}\Lambda} \sim R^0_{(\omega^*_N)^{l_1}\Lambda}, \tag{D.19}$$

manifestly true in view of, e.g., the invertibility of the elementary automorphism $\omega_{q,N}$. Accordingly, we next compare the eigenvalues of

$$M_{N+1,N+1} = k_{2\epsilon_{N+1}} \tag{D.20}$$

[26] The remaining one is the Casimir operator $\mathfrak{c}_1$.

[27] The more general equivalence: $R^{l_1}_{(\omega^*_N)^{l_1}\Lambda_1} \sim R^{l_2}_{(\omega^*_N)^{l_2}\Lambda_2}$ is ruled out upon noting that it would - in particular - require that the Casimir eigenvalues for $R^0_{\Lambda_1}$ and $R^0_{\Lambda_2}$ be equal, which is obviously not the case unless $\Lambda_1 = \Lambda_2$.

on $R^0_\Lambda$ and $R^l_{(\omega^*_N)^l\Lambda}$, $l \neq 0$. To these ends we take a general state of the module $\mathcal{H}_\Lambda$ and - respectively - of the module $\mathcal{H}_{(\omega^*_N)^l\Lambda}$ ($V_0, V_l$ are the corresponding highest weight states),

$$\mathcal{H}_\Lambda \ni v_{(m^0)} \underset{symb.}{\sim} F_1^{m^0_1} \dots F_N^{m^0_N} \triangleright V_0 \quad , \quad \mathcal{H}_{(\omega^*_N)^l\Lambda} \ni v_{(m^l)} \underset{symb.}{\sim} F_1^{m^l_1} \dots F_N^{m^l_N} \triangleright V_l, \tag{D.21}$$

with all $m^0_j, m^l_j \leq \sum_{i=1}^N \lambda_i$. Using (C.3) we then verify that

$$M_{N+1,N+1} \triangleright v_{(m^n)} = q^{h(n)} v_{(m^n)}, \quad n \in \{0, l\} \tag{D.22}$$

for

$$h(n) := 2m^n_N - \frac{2\kappa_N n}{N+1} - \frac{2s_n(\Lambda)}{N+1}, \tag{D.23}$$

in which

$$s_0(\Lambda) := \sum_{k=1}^N k\lambda_k, \tag{D.24}$$

$$s_l(\Lambda) := \sum_{k=1}^N k\omega^l_N(\lambda_k) = s_0(\Lambda) - (N+1)\sum_{k=l}^N \lambda_k + (N+1-l)\kappa. \tag{D.25}$$

In the present notation the condition of equivalence of the two irreducible representations boils down (in view of $m^L_N \leq \sum_{i=1}^N \lambda_i$) to the following statement:

$$\forall_{0\leq m^0_N \leq s'_0(\Lambda)} \exists_{0 \leq m^l_N \leq s'_l(\Lambda)} \; : \; m^l_N - m^0_N + \frac{s_0(\Lambda) - s_l(\Lambda)}{N+1} - \frac{l\kappa_N}{N+1} = 0 \mod \kappa_N, \tag{D.26}$$

where

$$s'_0(\Lambda) = \sum_{k=1}^N \lambda_k \quad , \quad s'_l(\Lambda) := \sum_{k=1}^N \left(\alpha_k, (\omega^*_N)^l\Lambda\right) = \kappa - \lambda_l. \tag{D.27}$$

An easy computation then shows the left hand side of (D.26) to be equal to

$$m^l_N - m^0_N + \sum_{k=l}^N \lambda_k - l - \kappa \tag{D.28}$$

and so, choosing[28] $m^0_N := \sum_{k=l}^N \lambda_k \leq s'_0(\Lambda)$ for an arbitrary $l \in \overline{1,N}$ and using $m^l_N \leq s'_l(\Lambda) \leq \kappa$, we arrive at the inequality:

$$-\kappa - l \leq (\text{D.28}) < 0. \tag{D.29}$$

Thus (D.26) is showed to be false which completes our proof of mutual inequivalence of the irreducible representations (D.18). $\Box$

[28] The existence of a nontrivial state for any such choice of $m^0_N$ follows from the Weyl symmetry of irreducible $\mathcal{U}_q(\mathfrak{su}(2))$-submodules of the given $\mathrm{REA}_q(A_N)$-module.

## References


[1] V. Schomerus, "Lectures on Branes in Curved Backgrounds", *Class.Quant.Grav.* **19** (2002) 5781, hep-th/0209241;

[2] E. Witten, "Nonabelian bosonization in two-dimensions", *Commun.Math.Phys.* **92** (1984) 455;

[3] D. Gepner, E. Witten, "String theory on group manifolds", *Nucl.Phys.* **B278** (1986) 493;

[4] C. Bachas, M. R. Douglas, C. Schweigert, "Flux Stabilization of D-branes", *JHEP* **05** (2000) 048, hep-th/0003037;
J. Pawełczyk, "$SU(2)$ WZW D-branes and their noncommutative geometry from DBI action", *JHEP* **08** (2000) 006, hep-th/0003057;
J. M. Maldacena, G. W. Moore, N. Seiberg,"D-Brane Instantons and K-Theory Charges", *JHEP* **11** (2001) 062, hep-th/0108100;
P. Bordalo, S. Ribault, C. Schweigert,"Flux Stabilization in Compact Groups", *JHEP* **10** (2001) 036, hep-th/0108201;

[5] A. Yu. Alekseev, V. Schomerus, "D-branes in the WZW model", *Phys.Rev.* **D60** (1999) 061901, hep-th/9812193;
G. Felder, J. Fröhlich, J. Fuchs, C. Schweigert, "The Geometry of WZW Branes", *J.Geom.Phys.* **34** (2000) 162, hep-th/9909030;

[6] N. Ishibashi,"The Boundary and Crosscap States in Conformal Field Theories", *Mod.Phys.Lett.* **A4** (1989) 251;
J. L. Cardy, "Boundary conditions, fusion rules and the Verlinde formula", *Nucl.Phys.* **B324** (1989) 581;
J. L. Cardy, D. C. Lewellen, "Bulk and boundary operators in conformal field theory", *Phys.Lett.* **B259** (1991) 274;
D. C. Lewellen, "Sewing constraints for conformal field theories on surfaces with boundaries", *Nucl.Phys.* **B372** (1992) 654;

[7] J. Pawełczyk, H. Steinacker, "Matrix description of D-branes on 3-spheres", *JHEP* **12** (2001) 018, hep-th/0107265;
J. Pawełczyk, H. Steinacker, "A quantum algebraic description of D-branes on group manifolds", *Nucl.Phys.* **B638** (2002) 433, hep-th/0203110;

[8] L. Dixon, J. A. Harvey, C. Vafa, E. Witten,"Strings on orbifolds", *Nucl.Phys.* **B261** (1985) 678;
L. Dixon, J. A. Harvey, C. Vafa, E. Witten,"Strings on orbifolds (II)", *Nucl.Phys.* **B274** (1986) 285;

[9] J. Fuchs, B. Schellekens, C. Schweigert, "From Dynkin diagram symmetries to fixed point structures", *Commun. Math. Phys.* **180** (1996) 39, hep-th/9506135;

[10] E. Sklyanin, "Boundary Conditions for Integrable Quantum Systems", *J. Phys.* **A21** (1988) 2375;

[11] P. P. Kulish, R. Sasaki, C. Schwiebert, "Constant Solutions of Reflection Equations and Quantum Groups", *J.Math.Phys.* **34** (1993) 286, hep-th/9205039;
P. P. Kulish, E. K. Sklyanin, "Algebraic Structures Related to Reflection Equations", *J.Phys.* **A25** (1992) 5963, hep-th/9209054;

P. P. Kulish, R. Sasaki, "Covariance Properties of Reflection Equation Algebras", *Prog.Theor.Phys.* **89** (1993) 741, hep-th/9212007;

[12] D. Arnaudon, M. Bauer, "Polynomial Relations in the Centre of $\mathcal{U}_q(sl(N))$", hep-th/9310030;

[13] S. Majid, "Examples of braided groups and braided matrices", *J.Math.Phys* **32** (1991) 3246;
S. Majid, *Foundations of Quantum Group Theory*, Cambridge University Press, 1995;

[14] K. Matsubara, V. Schomerus, M. Smedbäck, "Open Strings in Simple Current Orbifolds", *Nucl.Phys.* **B626** (2002) 53, hep-th/0108126;

[15] N. Kim, S.-J. Rey,"M(atrix) Theory on an Orbifold and Twisted Membrane", *Nucl.Phys.* **B504** (1997) 189, hep-th/9701139;
A. Konechny, A. Schwarz, "Compactification of M(atrix) theory on noncommutative toroidal orbifolds", *Nucl.Phys.* **B591** (2000) 667, hep-th/9912185;
A. Konechny, A. Schwarz, "Moduli spaces of maximally supersymmetric solutions on noncommutative tori and noncommutative orbifolds", *JHEP* **09** (2000) 005, hep-th/0005174;

[16] P. Podleś, "Quantum Spheres", *Lett. Math. Phys* **14** (1987) 193-202;

[17] P. M. Hajac, R. Matthes, W. Szymański, "Quantum Real Projective Space, Disc and Sphere", math.QA/0009185;

[18] N. Couchoud, "D-branes and orientifolds of $SO(3)$", *JHEP* **03** (2002) 026, hep-th/0201089;

[19] J. Pawełczyk, H. Steinacker, "Algebraic brane dynamics on $SU(2)$: excitation spectra", hep-th/0305226;

[20] G. Moore, N. Seiberg, "Classical and quantum conformal field theory", *Commun.Math.Phys.* **123** (1989) 177;

[21] A. N. Schellekens, S. Yankielowicz, "Extended chiral algebras and modular invariant partition functions", *Nucl.Phys.* **B327** (1989) 673;
A. N. Schellekens, S. Yankielowicz, "Simple currents, modular invariants and fixed points", *Int.J.Mod.Phys.* **A5** (1990) 2903;

[22] M. R. Douglas, B. R. Greene, D. R. Morrison, "Orbifold Resolution by D-Branes", *Nucl.Phys.* **B506** (1997) 84, hep-th/9704151;

[23] L. R. Huiszoon, K. Schalm, A. N. Schellekens, "Geometry of WZW Orientifolds", *Nucl.Phys.* **B624** (2002) 219, hep-th/0110267;

[24] J.Fuchs, *Affine Lie Algebras and Quantum Groups*, Cambridge University Press, 1992;

[25] A. Yu. Alekseev, A. Recknagel, V. Schomerus, "Non-Commutative World-volume Geometries: Branes on $SU(2)$ and Fuzzy Spheres", *JHEP* **09** (1999) 023, hep-th/9908040;
A. Yu. Alekseev, A. Recknagel, V. Schomerus, "Brane dynamics in background fluxes and non-commutative geometry", *JHEP* **05** (2000) 010, hep-th/0003187;

[26] L. Alvarez-Gaume, C. Gomez, G. Sierra, "Quantum group interpretation of some conformal field theories", *Phys.Lett.* **B220** (1989) 142;

[27] G. Moore, N. Reshetikhin, "A comment on quantum group symmetry in conformal field theory", *Nucl.Phys.* **B328** (1989) 557;

[28] H. Grosse, J. Madore, H. Steinacker, "Field Theory on the q-deformed Fuzzy Sphere I", *J.Geom.Phys.* **38** (2001) 308, hep-th/0005273;

[29] A. Klimyk, K. Schmüdgen, *Quantum Groups and Their Representations*, Springer-Verlag, 1998;

[30] N. Reshetikhin, M. A. Semenov-Tian-Shansky, "Central Extensions of Quantum Current Groups", *Lett. Math. Phys.* **19** (1990) 133-142;

[31] L. D. Faddeev, N. Yu. Reshetikhin, L. A. Takhtajan, "Quantization of Lie Groups and Lie Algebras", *Algebra Anal.* **1** (1989) 178;

[32] M. Noumi, "Mandonald's symmetric polynomials as zonal spherical functions on some quantum homogeneous spaces", math.QA/9503224.